

M4121, FINAL TEST

April 15–27, 2006.

Instructions: As usually, the main purpose of this test is to give you an opportunity to learn some new mathematics. You can use whatever books, papers or internet sources you like to dig out solutions, if you want. However, you can't just copy a solution from a book, your write-up should provide evidence that you understand what you are writing. Also, you are supposed to work on the test by yourself, clear evidence of collaboration in your solutions would result in a non-positive score. If you get too frustrated, you can always ask me for help – I might give you some hints. Don't hesitate to ask me if you are unsure what the problem asks for.

The test should be returned to me by the end of our last lecture at 11.30 on April, 27. Since ABSOLUTELY NO EXTENSIONS will be granted, I would suggest you to finish writing solutions by April 23.

1. (10 pt.) An easy one to warm up.

Suppose ν, ν_1, ν_2 are charges on (X, Σ) and μ is a measure on Σ . Prove the following:

- (a) If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $(\nu_1 + \nu_2) \perp \mu$.
- (b) If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $(\nu_1 + \nu_2) \ll \mu$.
- (c) If $\nu_1 \perp \nu_2$, then $|\nu_1| \perp |\nu_2|$.
- (d) $\nu \ll |\nu|$.
- (e) If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.

Use only the definitions in Chapters 3 and 8 of Bartle's book.

2. (20 pt.) Let's see if you have developed an intuition about measurability.

Give an example of two measurable sets $A, B \subset \mathbb{R}^2$ such that the set $A + B = \{z = x + y, x \in A, y \in B\}$ is not measurable. If you think that it is not possible, prove it.

3. (20 pt.) And now for your intuition about integrability.

- (a) Construct a continuous integrable function $f : \mathbb{R} \rightarrow \mathbb{R}_+$ such that $\limsup_{x \rightarrow +\infty} f(x) = +\infty$.
- (b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly continuous and integrable. Show that $\lim_{|x| \rightarrow +\infty} f(x) = 0$.

4. (20 pt.) Let's see if you can learn some abstract probability.

Let (X, Σ, μ) be a σ -finite measure space, and let $f : X \rightarrow \mathbb{R}$ be an integrable (and, hence, Σ -measurable) function. Let $\Sigma_0 \subset \Sigma$ be a smaller σ -algebra. Give an example, when f is not Σ_0 -measurable. Using Radon-Nikodym Theorem show that, nevertheless, there exists

a unique Σ_0 -measurable function f_0 such that $\int fg d\mu = \int f_0 g d\mu$ for every Σ_0 -measurable g for which the integrals are finite. The function $f_0 = E(f | \Sigma_0)$ is called the *conditional expectation* of f with respect to Σ_0 . Explain in detail why this name is appropriate.

5. (30 pt.) Finally, the promised transformation problem.

Let E be a measurable set in \mathbb{R}^d and T a linear transformation of \mathbb{R}^d to \mathbb{R}^d . Show that $T(E)$ is measurable and

$$m(T(E)) = |\det T| m(E),$$

where m is the Lebesgue measure. Proceed as follows.

- (a) To show that $T(E)$ is measurable, prove first that if E is an F_σ set, so is $T(E)$. Then use the inequality $|T(x) - T(y)| \leq \|T\| |x - y|$ to show that $m(E) = 0$ implies $m^*(T(E)) < c\epsilon$ for any $\epsilon > 0$ and, hence, $m(T(E)) = 0$. Finally, use Corollary 15.8 on p. 159 of Bartle's book to conclude that $T(E)$ is measurable. You do not need to prove the corollary.
- (b) To obtain the qualitative result, we will use the so-called "LDU"-decomposition of T , that is we will let $T = LDU$, where L and U are, respectively, lower and upper triangular matrices with ones on the main diagonal and D is a diagonal matrix. Such a decomposition always exists for a suitable permutation of T , this is a fact from matrix theory which you do not need to prove. Since $\det T = \det L \cdot \det D \cdot \det U$ it is enough to prove the result for each class of matrices separately. For D the result should be pretty straightforward. The proof for L and U will, of course, be similar. First, consider the case $d = 2$ and the upper triangular transformation U given by $x' = x + ay$, $y' = y$. Then

$$\chi_{U(E)}(x, y) = \chi_E(U^{-1}(x, y)) = \chi_E(x - ay, y), \quad \text{and, hence,}$$

$$m(U(E)) = \int \chi_E(x - ay, y) dm = \int \chi_E(x, y) dm = m(E)$$

by the translation invariance of the measure. You do not need to prove translation invariance but you do need to explain why it can be used in this way. Alternatively, you can use the polar decomposition of a matrix.