M4111, FINAL TEST
December 6–14, 2005.

Instructions: The main purpose of this test is to give you an opportunity to `learn` *some nice mathematics in a not too stressful atmosphere. For this reason, I suggest you to forget about grades (I will introduce some curving, if necessary) and start having fun doing math. Because solving these problems is supposed to be fun some of them are somewhat difficult. Some others are difficult but standard – you can use whatever books, papers or internet sources you like to dig out solutions, if you want (of course, you can't just copy a solution from a book, your write-up should provide evidence that you understand what you are writing). You are supposed to work on the test by yourself but you can always ask me for help. I might give you some hints, although I will be less inclined to do it than usually.*

The test should be returned to me or to Cupples I, Room 100 by 4.00 PM, December 14.

1. An easy one to start with.

   Suppose that a function $f : U \subset \mathbb{R} \to \mathbb{R}$ is continuous, bounded and monotonic, and U is an open interval (or an open half-line). Prove that f is uniformly continuous.

   Solution. For a bounded interval the result follows (almost) immediately from the monotone convergence theorem and Problem 4 in Homework 7. For an unbounded interval on can use a "dirty trick" of extending the real line by $\{\infty\}$ and $\{-\infty\}$. The best way of solving the problem is the following direct argument. For any given $\epsilon > 0$ break the (bounded) interval $[\inf f(U), \sup f(U)]$ into finitely many equal-size subintervals I_j, $j = 1, \dots, J$, of length less than $\epsilon/2$. Let $\delta > 0$ be less than the minimal length of the intervals $f^{-1}(I_j)$, $j = 1, \dots, J$ (these are intervals because f is monotonic). Then, clearly, $|f(x) - f(y)| < \epsilon$ when $|x - y| < \delta$.

   I'm surprised that only three of you provided a complete solution of this problem. Only one of the three gave (essentially) the argument above. For this reason the maximal score for this problem ended up being almost twice higher than for each of the following two problems.

2. This one is longer but very standard. There are a number of ways of doing it. I would suggest to use the one(s) for which all the prerequisites were covered during the course (otherwise you should develop them in your solution).

Prove that the limit $e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$, $n \in \mathbb{N}$, is a well-defined number between $\frac{8}{3}$ and 3. Do a little bit more work to show that $e = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$, $x \in \mathbb{R}$.

Solution. Many of you misunderstood the statement of the problem. All you needed to do was to rigorously prove that the above limits exist and are equal to each other. You did NOT have to prove that they are equal to $\sum_{n=0}^{\infty} \frac{1}{n!}$.

Consider the sequence $a_n = \left(1+\frac{1}{n}\right)^n$, $n = 1, 2, \ldots$. We shall prove that it is increasing and bounded above. Then Monotone convergence theorem would imply that the sequence converges.

To prove boundedness we use binomial expansion of a_n:

$$a_n = 2 + \frac{n-1}{n} \cdot \frac{1}{2!} + \frac{(n-1)}{n} \cdot \frac{n-2}{n} \cdot \frac{1}{3!} + \cdots + \frac{(n-1)!}{n^{n-1}} \cdot \frac{1}{n!} <$$

$$2 + \sum_{m=1}^{n-1} 2^{-m} < 2 + \sum_{m=1}^{\infty} 2^{-m} = 3.$$

The sequence is increasing because the binomial expansion for a_{n+1} –

$$a_{n+1} = 2 + \left(1 - \frac{1}{n+1}\right) \cdot \frac{1}{2!} + \left(1 - \frac{1}{n+1}\right) \cdot \left(1 - \frac{2}{n+1}\right) \cdot \frac{1}{3!} + \cdots$$

$$+ \left(1 - \frac{1}{n+1}\right) \cdots \left(1 - \frac{n-1}{n+1}\right) \cdot \frac{1}{n!} + \frac{n!}{(n+1)^n} \cdot \frac{1}{(n+1)!}$$

– is easily seen to be bigger than that of a_n.

Observe that $a_{29} > \frac{8}{3}$ ensures that e is indeed within the specified limits. (Sorry, I thought, I had $\frac{7}{3}$ in which case a_4 is enough).

For the "continuous" limit, let $f(x) = \left(1+\frac{1}{x}\right)^x$, $x \in \mathbb{R}$. Since $f'(x) = \frac{d}{dx}\left(e^{x\ln\left(1+\frac{1}{x}\right)}\right) = \left(\ln\left(1+\frac{1}{x}\right) + \frac{x^2}{1+x}\right)\left(1+\frac{1}{x}\right)^x > 0$ when $x > 0$, the function f is increasing on $\mathbb{R}_+$. Now the Squeeze theorem together with the three propositions on p. 74 and p. 46 yields the desired result. Note that the Monotone convergence theorem does NOT (directly) apply because we only proved it for sequences, not for functions.

3. Here is another standard problem about the exponential function which is pertinent to many branches of analysis. It has a very elegant solution

using the fundamental theorem of calculus and uniqueness theorem for the ODE solutions. However, since we haven't covered those in class, you will probably end up with a standard solution.

Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous function which satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}_+$. Prove that $f(x) = a^x$, for all $x \in \mathbb{R}_+$, where $a = f(1)$.

By the way, an analogous result remains true in a much more general situation, say, for matrix-valued functions.

Solution. Most of you presented the standard solution which consists of proving the result first for natural numbers, second for integer and rational numbers and, finally, extending by continuity to all of $\mathbb{R}$. Not many of you presented a flawless argument but usually it was close enough. For this reason, I will only sketch the proof based on the FTC mentioned above. Indeed, the uniqueness theorem for the ODE solutions implies that all we need to do is to show that continuity of f implies its differentiability. Also $f(0) = f(0)^2$ implies that $f(0) = 1$ or $f(0) = 0$. In the latter case there is nothing to prove because then $f \equiv 0$. If $f(0) = 1$, consider the following antiderivative of f (it is well-defined because f is continuous): $F(x) = \int_0^x f(t)dt$. Observe that since f is easily seen to be non-negative and $f(0) = 1 > 0$, $F(x) > 0$ when $x > 0$. This and

$$F(x+\delta) = \int_0^{x+\delta} f(t)dt = \int_0^x f(t)dt + \int_x^{x+\delta} f(t)dt =$$

$$F(x) \int_0^{\delta} f(t+x)dt = F(x)F(\delta)f(x)$$

imply that $f(x) = \frac{F(x+\delta)}{F(x)F(\delta)}$ is a differentiable function.

It is worth noting that there are discontinuous functions satisfying the above functional equation. The easiest of them is $f = \chi_{\{0\}}$.

4. Now it's time to have some fun. I hope, the economics students will appreciate the problem.

   An "aggregated" daily diet of a postdoc consists of certain number of servings of milk (x_1), bread (x_2), meat (x_3), fish (x_4), and fruit (x_5). The associated costs per serving (c_i) and calories per serving (k_i) are in the table below. Assuming that the postdoc does not want to starve (i.e. consume less than 1800 calories a day), grow fat (eat more than 3000 calories a day), or get broke (spend more than 30$ a day), compute the diet which maximizes the "taste" function

$$f(x) = \ln((x_2^2 - x_1^2)(x_3^2 - x_4^2)x_5^3) + 0.8x_1 - 1.2x_2 - \frac{20}{17}x_3 + \frac{14}{17}x_4 - \frac{x_5^3}{\pi^3}.$$

i	1	2	3	4	5
c_i	.5	1	3	3	2.5
k_i	130	150	200	70	110

Is the postdoc satisfied with his financial situation (foodwise)?

A few comments. The number of servings does not have to be integer (otherwise the problem would've been more difficult and would have very little to do with calculus). The chosen data make the problem easier than it could have been. You may want to use the second derivative test in the solution.

This kind of problems became quite popular in the Soviet Union in the early sixties. At that time nobody cared about taste functions trying just to minimize the cost and survive. The first solutions were stunning – they suggested to sustain the population on sunflower oil and vinegar. Quite a drastic difference compared to certain modern food manufacturers hiring engineers to compute the optimal friction in the mouth induced by consumption of their product.

Comments. This problem was designed in part to illustrate that sometimes problems that look hard may, in fact, be solved by very elementary methods. Then using Lagrange Multipliers or Kuhn-Taker Theorem can be compared to cannon-hunting the cockroaches. Another illustrative point of this problem is the fact that just one simple condition (which I meant to but did not include) can make a difference between an extremely easy and a very difficult problem. In this particular case, the condition is: "The postdoc prefers to eat more servings of meat than fish (*i.e.*, $x_3 > x_4$)".

The catch in this problem which all of us overlooked was the fact that the local maximum most of you found was NOT the global maximum. It would have been under the condition $x_3 > x_4$. Below you will see an easy way of proving it.

We need to maximize $f = f(x_1, x_2, x_3, x_4, x_5)$ subject to constraints

$$\sum_{i=1}^{5} c_i x_i \leq 30; \quad 1800 \leq \sum_{i=1}^{5} k_i x_i \leq 3000; \quad x_i \geq 0, \; i = 1, \ldots, 5.$$

Let us find the critical point of f. By equating first partials of f to 0 we get

$$\frac{2x_1}{x_2^2 - x_1^2} = 0.8;\ \frac{2x_2}{x_2^2 - x_1^2} = 1.2;\ \frac{2x_3}{x_3^2 - x_4^2} = \frac{20}{17};\ \frac{2x_4}{x_3^2 - x_4^2} = \frac{14}{17};\ x_5 = \pi.$$

Simplifying, we see that

$$x_1 + x_2 = 5;\ x_2 - x_1 = 1;\ x_3 + x_4 = \frac{17}{3};\ x_3 - x_4 = 1;\ x_5 = \pi,\ \text{or}$$

$$x_1 = 2;\ x_2 = 5;\ x_3 = \frac{10}{3};\ x_4 = \frac{7}{3};\ x_5 = \pi.$$

It is immediate that this critical point satisfies the calories and budget constraints. To examine this critical point let us change variables as suggested by one of the lines above:

$$u_1 = x_1 + x_2;\ u_2 = x_2 - x_1;\ v_1 = x_3 + x_4;\ v_2 = x_3 - x_4.$$

Under this change of variables our function f changes to

$$g(u_1, u_2, v_1, v_2, x_5) = g_1(u_1) + g_2(u_2) + g_3(v_1) + g_2(v_2) + g_5(x_5) =$$

$$\ln u_1 - 0.2u_1 + \ln|u_2| - u_2 + \ln v_1 - \frac{3}{17}v_1 + \ln|v_2| - v_2 + 3\ln x_5 - \frac{x_5^3}{\pi^3}.$$

Under the condition $x_3 > x_4$ the absolute values in the above formula can be erased and the usual techniques of SINGLE variable calculus tell us that our critical point is, indeed, the ABSOLUTE maximum on this (smaller) domain. However, if we allow $x_3 < x_4$, the point $x = (4.5, 1.5, 1.5, 4.5, 1)$, for example, satisfies our constraints and yields a bigger value of f than the local maximum found above. At this point, some fancy techniques (mentioned above) or some hard work from first principles is needed to find the solution (it's relatively easy to prove that it exists, but much harder to find it). Since this is not what I wanted you to do I will spare you the time and myself the effort of actually doing it.

5. Back to work. When we proved Taylor's theorem in class we used the remainder in the form due to Lagrange. There are several others which maybe of use. In this problem you will develop some of them. To make our life easier we will do that for single variable functions.

Prove the following theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be $n+1$ times differentiable. Then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where, given any $p > 0$,

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{n!p}(x-\xi)^{n+1-p}(x-a)^p,$$

for some ξ between x and a. This form is sometimes called Schlömilch-Roche Remainder. The choices $p = n+1$ and $p = 1$ give the Lagrange and Cauchy remainders, respectively.

Assuming that f is only n times differentiable prove that $R_n(x) = o(|x-a|^n)$. This form of the remainder is sometimes attributed to Peano.

As an icing, compute R_5 for the Taylor series at 0 of $f(x) = e^{-\frac{1}{x^2}}$ with $f(0) = 0$.

Solution. All you needed to do to prove the Schlömilch-Roche formula was to go through the proof of the Taylor's theorem on p.107 of your text, make the following two changes:

$$R_n(b, a) = K\frac{(b-a)^p}{(n+1)!} \text{ and } \varphi(x) = R_n(b, x) - K\frac{(b-x)^p}{(n+1)!},$$

and proceed from there. I was amazed how much of notational mess some of you created in this argument. Few of you managed to grasp the distinction between R_n as a function of one variable (as used in the statement of the problem) and a function of two variables denoted by the same symbol in your textbook.

Peano's formula is an easy consequence of Lagrange formula for R_{n-1}. Indeed,

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!}(x-a)^n - \frac{f^{(n)}(a)}{n!}(x-a)^n = \frac{f^{(n)}(\xi) - f^{(n)}(a)}{n!}(x-a)^n = o(|x-a|^n)$$

by continuity of $f^{(n)}$. Thus, no L'Hospital's rule is needed in the proof.

Finally, for $f(x) = e^{-\frac{1}{x^2}}$, it is easy to see that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Hence, $R_n \equiv f$, $n \in \mathbb{N}$. This is a standard example of a function f whose Taylor series converges everywhere but not to f.

6. Now it is time to check what you can do with the Implicit Function Theorem. I did something very similar in class.

   For $j = 1, 2, 3$ let $f_j(x, y, z) = x^j + y^j + z^j$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F = (f_1, f_2, f_3)$.

   (a) Compute the Jacobian matrix $F'(x_0, y_0, z_0)$ and show that it is invertible if and only if x_0, y_0, and z_0 are distinct numbers (no two are equal). Deduce a statement about the existence of differentiable local inverses of F.

   (b) For $(x_0, y_0, z_0) = (-1, 0, 1)$ and G a local inverse of F from an open neighborhood of $(u_0, v_0, w_0) = F(x_0, y_0, z_0)$ back to an open neighborhood of (x_0, y_0, z_0), compute the affine approximation of $G(u, v, w)$ about (u_0, v_0, w_0).

   (c) For arbitrary (x_0, y_0, z_0), let $v_0 = f_2(x_0, y_0, z_0)$ and $w_0 = f_3(x_0, y_0, z_0)$. Give necessary and sufficient conditions on (x_0, y_0, z_0) under which the two simultaneous equations

   $$f_2(x, y, z) = v_0, \quad f_3(x, y, z) = w_0$$

   can be solved for two of the three variables x, y, z as continuously differentiable functions of the third variable.

Solution.

(a) Direct computation shows that

$$F'(x_0, y_0, z_0) = \begin{pmatrix} 1 & 1 & 1 \\ 2x_0 & 2y_0 & 2z_0 \\ 3x_0^2 & 3y_0^2 & 3z_0^2 \end{pmatrix} \quad \text{and}$$

$$\det F'(x_0, y_0, z_0) = 6 \begin{vmatrix} 1 & 1 & 1 \\ x_0 & y_0 & z_0 \\ x_0^2 & y_0^2 & z_0^2 \end{vmatrix} = 6(y_0 - x_0)(z_0 - x_0)(z_0 - y_0).$$

More knowledgeable of you noticed that the above determinant is a Vandermonde determinant and used the corresponding formula circumventing the otherwise lengthy computation. Since a matrix is invertible iff its determinant is non-zero the statement follows and the Inverse function theorem applies: for $(x_0, y_0, z_0) \in \mathbb{R}^3$ with $x_0 \neq y_0$, $x_0 \neq z_0$, and $z_0 \neq y_0$, there exist open subsets $U, V \in \mathbb{R}^3$ such that $(x_0, y_0, z_0) \in V$ and the restriction of F to V is a 1-1 map of V onto U whose inverse $F^{-1} : U \to V$ is continuously differentiable.

(b) Surprisingly, but quite a few of you messed up the formula for the affine approximation (tangent line): $G(u,v,w) = G(u_0,v_0,w_0) + G'(u_0,v_0,w_0)(u-u_0, v-v_0, w-w_0)^T$. By the inverse function theorem, we get

$$G(u,v,w) = (x_0,y_0,z_0)^T + [F'(x_0,y_0,z_0)]^{-1}(u-u_0, v-v_0, w-w_0)^T.$$

Finally, plugging in the given point, we get

$$G(u,v,w) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 0 & -3 & 2 \\ 12 & 0 & -4 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} u \\ v-2 \\ w \end{pmatrix}.$$

(c) The catch here was to check if you realized that the Implicit Function Theorem gives only sufficient conditions not necessary and sufficient. Since we didn't discuss the necessary conditions in class, I wasn't expecting you to provide them. However, I was hoping that more of you will notice the difference.

Anyway, as many of you figured out but few completed the computation, the sufficient conditions can be read off the statement of the IFT. Say, x and y can be expressed (locally) as continuously differentiable functions of z if

$$\begin{vmatrix} \frac{\partial f_2}{\partial x}(x_0,y_0,z_0) & \frac{\partial f_2}{\partial y}(x_0,y_0,z_0) \\ \frac{\partial f_3}{\partial x}(x_0,y_0,z_0) & \frac{\partial f_3}{\partial y}(x_0,y_0,z_0) \end{vmatrix} = \begin{vmatrix} 2x_0 & 2y_0 \\ 3x_0^2 & 3y_0^2 \end{vmatrix} = 6x_0y_0(y_0-x_0) \neq 0,$$

that is, if $x_0 \neq y_0$ and $x_0y_0 \neq 0$.

7. Construction of fractal sets via so-called iteration function systems relies on a metric space $\mathcal{K} = \mathcal{K}(\mathbb{R}^n)$ of non-empty compact subsets of $\mathbb{R}^n$. This space turns out to be a complete metric space with respect to the Hausdorff metric h which is defined as follows. For $X \subset \mathbb{R}^n$ let $B_r(X) = \{y \in \mathbb{R}^n : d(y,X) < r\}$, where d is the usual Euclidean metric in $\mathbb{R}^n$. Then, for $X, Y \in \mathcal{K}$,

$$h(X,Y) = \inf\{r \in \mathbb{R} : X \subset B_r(Y) \,\text{and}\, Y \subset B_r(X)\}.$$

Show that h is indeed a metric on $\mathcal{K}$.

Now let A be an invertible $n \times n$ matrix such that $d(Ax,0) \leq kd(x,0)$ for all $x \in \mathbb{R}^n$ and some $k < 1$. Let also Λ be a finite set of vectors with integer entries. Assuming that $(\mathcal{K}, h)$ is complete (this is not so easy to prove), show that the equation

$$X = \bigcup_{\ell \in \Lambda} A(X + \ell)$$

has a unique solution in $\mathcal{K}$. Such a solution can be a simple parallelogram or a complicated fractal set, *e.g.*, the "twin dragon".

Comments. This problem, obviously, was designed to check how much you understood of the ABSTRACT metric spaces and maps between them rather than just functions on Euclidean spaces. Unfortunately, many of you are still wrestling to understand the difference between, say, a unit ball in $\mathbb{R}^n$ and a unit ball in the space of bounded continuous functions on a compact set.

Anyway, checking that h is a well-defined metric on $\mathcal{K}$ is a routine exercise. Here is what many of you forgot or failed to do:

- Check that h is well-defined for all sets in $\mathcal{K}$. Indeed, if unbounded or empty sets were aloud, this would not be true (think of an example or find it in the internet).
- Check that $h(X, Y) = 0$ implies $X = Y$. Indeed, if the sets that are not closed were allowed, this would not be true. Very few of you provided a rigorous proof of this statement. An instructive proof goes by contrapositive.
- Rigorously prove the triangle inequality. It does follow from the triangle inequality for the Euclidean metric but not immediately.
- Be careful with the definition of the inf. Indeed, $h(X, Y) = r$ does not, a priori, mean that $X \subseteq B_r(Y)$! It only means that $X \subseteq B_{r+\epsilon}(Y)$ for every $\epsilon > 0$.

In the second part of the problem you were to check that the fixed point theorem applies for the mapping

$$\omega : X \mapsto \bigcup_{\ell \in \Lambda} A(X + \ell).$$

The first thing you should have mentioned was that ω does map $\mathcal{K}$ into $\mathcal{K}$. I did not penalize those who could not check it but did subtract a point for not mentioning it. The next (and last) thing was to check that ω (NOT A!) is a contraction, that is, for all $X, Y \in \mathcal{K}$,

$$h(\omega(X), \omega(Y)) \leq \kappa h(X, Y) \quad \text{for some} \quad \kappa < 1.$$

This follows almost immediately from

$$y \in B_r(X) \Rightarrow d(y, X) < r \Rightarrow d(Ay, AX) < kr \Rightarrow$$

$$d\left(Ay + \ell, \bigcup_{\ell \in \Lambda} A(X + \ell)\right) < kr,$$

where we used linearity of A.