

Homework Sets

Math 4111, Fall '05, I. Krishtal

WARNING: “Solutions” given here are, in general, not complete. Some of them would be acceptable if you encountered similar problems in another (more advanced) course but for this course YOUR solutions should be more detailed. If you are using these to prepare for the test it would be a good idea to check that you can supply all the tiny details which I did not spell out here. Solutions below is a Guide, not a Bible.

Homework 12

Problems for all.

Solve problems 8,12,13,15,38 on pp. 161-166 and 19 on p. 214.

Extra problems.

Solve problems 34, 41 on pp. 165-167

Solutions.

Problem 8.

(a) Pick an arbitrary $\epsilon > 0$. Find N such that for every $m \geq N$ we have $\sum_{k=m}^{\infty} a_k < \epsilon/2$.

Then for every $n > 2N$

$$na_n < 2 \sum_{k=N}^n a_k < \epsilon.$$

(b) Follows by comparison from

$$\sum_{n=1}^{\infty} a_n \leq \sum_{k=0}^{\infty} 2^k a_{2^k} \leq 2 \sum_{n=1}^{\infty} a_n.$$

Problem 12.

Since $\lim_{n \rightarrow \infty} a_n = 0$, there exists $N \in \mathbb{N}$ such that $|a_n|^2 < |a_n| < 1$ for all $n > N$. The result now follows by comparison.

Problem 13.

Follows by comparison with the geometric series $\sum \rho^n$.

Problem 15.

First of all $\sum_{m,n=1}^{\infty} a_n b_m$ converges absolutely. This follows because its partial sums in any order are bounded above by products of partial sums of $\sum a_n$ and $\sum b_m$. The equality follows from the proposition on p.148 and

$$\left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{m=1}^{\infty} b_m \right) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} b_m \right) a_n = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} a_n b_m \right).$$

Problem 38.

Taking log, passing to a continuous limit, and applying L'Hospital's rule, we get

$$\begin{aligned} \lim_{x \rightarrow +\infty} x \ln \frac{a^{\frac{1}{x}} + b^{\frac{1}{x}} + c^{\frac{1}{x}}}{3} &= \lim_{x \rightarrow 0^+} \frac{\ln \frac{1}{3}(a^x + b^x + c^x)}{x} = \\ \lim_{x \rightarrow 0^+} \frac{a^x \ln a + b^x \ln b + c^x \ln c}{a^x + b^x + c^x} &= \frac{1}{3}(\ln a + \ln b + \ln c). \end{aligned}$$

Hence, the original limit is equal to $(abc)^{\frac{1}{3}}$.

Problem 19.

Assuming that the partials f_x , f_y , and f_z are all continuous and non-zero, we can apply implicit function theorem to obtain 3 functions $\phi_x = \phi_x(y, z)$, $\phi_y = \phi_y(x, z)$, and $\phi_z = \phi_z(x, y)$. We then deduce

$$\frac{\partial \phi_z}{\partial y} \cdot \frac{\partial \phi_y}{\partial x} \cdot \frac{\partial \phi_x}{\partial z} = -1$$

from $0 = f_x + f_y \cdot \frac{\partial \phi_y}{\partial x} = f_y + f_z \cdot \frac{\partial \phi_z}{\partial y} = f_z + f_x \cdot \frac{\partial \phi_x}{\partial z}$.

Homework 11

Problems for all.

Solve problems 8,12,16 on pp. 213-214 and 2-3 on pp. 190-191.

Solutions.

Problem 8.

This is part of the d'Alembert's formula for the 1-d wave equation. Indeed, the functions φ and ψ represent the so-called traveling waves. To prove the result observe that the change of variables $u = x - ay$ and $v = x + ay$ transforms the original equation to $4\frac{\partial^2 g}{\partial u \partial v} = 0$ which is clearly solved by $g(u, v) = \varphi(u) + \psi(v)$.

Problem 12.

When $(x, y) \neq (0, 0)$, $\frac{\partial f}{\partial x}(x, y) = \frac{x^2 y(x^2 + 3y^2)}{(x^2 + y^2)^2}$ and $\frac{\partial f}{\partial y}(x, y) = \frac{x^3(x^2 - y^2)}{(x^2 + y^2)^2}$. At the origin, we have

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \rightarrow 0} 0/x = 0 = \lim_{y \rightarrow 0} 0/y = \frac{\partial f}{\partial y}(0, 0).$$

Next, we show that these partial derivatives are continuous. Observe that the existence of the second partials which we will show later does NOT, in general, imply continuity of the first partial derivatives. However, it does follow from the following inequalities:

$$\frac{x^2|y|(x^2 + 3y^2)}{(x^2 + y^2)^2} \leq \frac{3x^2|y|}{x^2 + y^2} \leq \frac{3/2|x|(x^2 + y^2)}{x^2 + y^2} \leq \frac{3}{2}|x|;$$

$$\frac{|x|^3(x^2 - y^2)}{(x^2 + y^2)^2} \leq \frac{|x|(x^2 + y^2)^2}{(x^2 + y^2)^2} \leq |x|.$$

Thus, f is continuously differentiable. Let us compute the second partials at the origin.

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = \lim_{x \rightarrow 0} 0/x = 0 \text{ and } \frac{\partial^2 f}{\partial y^2}(0, 0) = \frac{\partial^2 f}{\partial x \partial y}(0, 0) = \lim_{y \rightarrow 0} 0/y = 0.$$

$$\text{However, } \frac{\partial^2 f}{\partial y \partial x}(0, 0) = \frac{d}{dx}x = 1.$$

Problem 16.

Let $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be differentiable on an open set U and attain maximum (or minimum) at $a \in U$. Then single variable functions $f_i(x_i) = f(a_1, \dots, x_i, \dots, a_n)$ are differentiable and attain maximum (or minimum) at a_i . Hence, $0 = f'_i(a_i) = \frac{\partial f}{\partial x_i}(a)$.

Assume now that f is twice differentiable, all first partials are 0 at a and the Hessian $H(a)$ (the matrix of second partials at a) is positive definite, that is

$$(H(a)x, x) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a)x_i x_j \geq c \sum_{i=1}^n x_i^2 \text{ for all } x \in \mathbb{R}^n.$$

Now it's time to use Taylor's formula. However, we have to be careful with the remainder term because we cannot assume that our function is 3 times differentiable. Fortunately,

since all second partials are continuous, we have $f(x) = f(a) + 0 + \frac{1}{2}(H(\xi)(x-a), (x-a)) = f(a) + \frac{1}{2}(H(a)(x-a), (x-a)) + \frac{1}{2}((H(\xi) - H(a))(x-a), (x-a)) = f(a) + \frac{1}{2}(H(a)(x-a), (x-a)) + o(\|x-a\|^2)$. This implies

$$f(x) \geq f(a) + \frac{c}{2}\|x-a\|^2 + o(\|x-a\|^2) \geq f(a)$$

in some open set around a . Hence, a is a local minimum. Analogously, if $H(a)$ is negative definite, a is a local maximum.

If $H(a)$ is neither positive semidefinite nor negative semidefinite, in every open neighborhood of a there exist z and y such that $(H(a)(z-a), (z-a)) < 0$ and $(H(a)(y-a), (y-a)) > 0$. Then our slightly refined Taylor's formula

$$f(x) = f(a) + 0 + \frac{1}{2}(H(a)(x-a), (x-a)) + o(\|x-a\|^2)$$

precludes a from being a local extremum.

Problem 2.

This is the same Babylonian method we discussed in the proof of Weierstrass theorem. Let $f(x) = x^2 - a$. By Newton's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n \in \mathbb{N}$. Since $f'(x) = 2x$, we obtain $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}\frac{a}{x_n} = \frac{1}{2}(x_n + \frac{a}{x_n}) =: g(x_n)$. Obviously, if the limit L exists, it has to satisfy $L^2 = a$. Moreover, if $x_0 > 0$ the limit cannot be negative. Moreover, in this case, $x_1 = g(x_0) = \frac{1}{2}(x_0 + \frac{a}{x_0}) = \frac{1}{2}(2\sqrt{a} + (\sqrt{x_0} - \sqrt{\frac{a}{x_0}})^2) \geq \sqrt{a}$. Hence, $g((0, \infty)) \subset [\sqrt{a}, \infty)$. Since $g'(x) = \frac{1}{2}(1 - \frac{a}{x^2}) < \frac{1}{2} < 1$ and is positive when $x > \sqrt{a}$, the function g is a contraction and the fixed point theorem applies.

Problem 3.

We are after the fixed point of the function $f(x) = \cos x - \frac{1}{2}$ on $[0, \frac{\pi}{4}]$. First, $f([0, \frac{\pi}{4}]) \subset [0, 1] \subset [0, \frac{\pi}{4}]$. Moreover, since $f'(x) = -\sin x$, $|f'(x)| \leq \frac{1}{\sqrt{2}}$ on $[0, \frac{\pi}{4}]$, and, by the proposition on p. 172, f is a contraction. Hence, the fixed point exists and is unique on $[0, \frac{\pi}{4}]$. The last interesting point in this problem is to figure out when to stop iterations being sure that the needed accuracy has been achieved. Observe that the contraction property implies that $|x_n - L| \leq k|x_{n-1} - L| \leq \dots \leq k^n|x_0 - L| \leq \frac{\pi}{2^{2+n/2}}$. Hence, we can get the following pairs (estimate; error): $(0; \frac{\pi}{4})$, $(0.5; 0.55536)$, $(0.37758; 0.39270)$, $(0.42956; 0.27768)$, $(0.40915; 0.19635)$, $(0.41746; 0.13884)$, $(0.41412; 0.09817)$, $(0.41547; 0.06942)$, $(0.41493; 0.04909)$, $(0.41515; 0.03471)$, $(0.41506; 0.02454)$, $(0.41509; 0.01736)$... At this point we seem to have achieved the desired accuracy, although our error estimate remains too big. In fact, we can make a much better estimate in this particular case. Observe that our function f is (strictly) decreasing on $[0, \frac{\pi}{4}]$ while the identity function is strictly increasing. This immediately implies that if $x_n > L$ then $x_{n+1} < L$ and if $x_n < L$ then $x_{n+1} > L$. Therefore, $|x_n - L| < |x_{n+1} - x_n|$. That means that we already have our solution correct upto 4 decimal places.

Homework 10

Problems for all.

1. Give an example of a real-valued differentiable function of 2 variables which is not continuously differentiable.

Solution. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x^2 \sin \frac{1}{x}$ (with $f(0, 0) = 0$). Indeed,

$$0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{|f(x, y)|}{\sqrt{x^2 + y^2}} \leq \lim_{x \rightarrow 0} \frac{|x^2 \sin \frac{1}{x}|}{|x|} = 0$$

implies that f is differentiable at 0 but we know that $\frac{\partial f}{\partial x}$ is not continuous. A “non-separable” example is provided by a linear change of variable.

2. Find a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is not continuous at 0 but has the directional derivative in the direction of the vector $(1, 1)$.

Solution. We know that the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = \frac{xy}{x^2 + y^2}$ (with $f(0, 0) = 0$) is not continuous at the origin but the partial derivatives exist. We need to rotate this function by 45 degrees (in either direction). This is done by the change of variables. We get $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g(u, v) = \frac{u^2 - v^2}{u^2 + v^2}$. Let us compute the partial for the sake of completeness:

$$(D_{\mathbf{u}}g)(u, v) = \lim_{t \rightarrow 0} \frac{g(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}})}{t} = \lim_{t \rightarrow 0} \frac{0}{t^3} = 0.$$

3. Let $f(x, y, z) = (x + y + z, x^2 + y^2 + z^2)$ and $g(t, s) = (e^{ts}, e^{t-s})$. Find the (matrix) derivative of $g \circ f$, first, directly by substituting and, second, by using the chain rule.

To make things shorter let $h = g \circ f$, $u(x, y, z) = e^{(x+y+z)(x^2+y^2+z^2)}$, and $v(x, y, z) = e^{(x+y+z)-(x^2+y^2+z^2)}$. Then $h(x, y, z) = (u(x, y, z), v(x, y, z))$ and $h'(x, y, z) =$

$$\left(\begin{array}{ccc} (x^2 + (x + y)^2 + (x + z)^2)u(x, y, z) & \dots & \dots \\ (1 - 2x)v(x, y, z) & (1 - 2y)v(x, y, z) & (1 - 2z)v(x, y, z) \end{array} \right),$$

where the omitted term are obtained by the obvious symmetry. Alternatively,

$$f'(x, y, z) = \begin{pmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \end{pmatrix} \quad \text{and}$$

$$g'(f(x, y, z)) = \begin{pmatrix} (x^2 + y^2 + z^2)u(x, y, z) & (x^2 + y^2 + z^2)u(x, y, z) \\ v(x, y, z) & -v(x, y, z) \end{pmatrix}.$$

Clearly, $h'(x, y, z) = g'(f(x, y, z)) \cdot f'(x, y, z)$

4. Solve problem 1 on page 212.

Solution. The function is clearly continuous when $(x, y) \neq (0, 0)$. Let us show that it is continuous at the origin as well: $\lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{|x|+|y|} \leq \lim_{(x,y) \rightarrow (0,0)} \frac{|xy|+x^2}{|x|+|y|} = \lim_{x \rightarrow 0} |x| = 0.$

It is clear that the function is differentiable when $xy \neq 0$. When $xy = 0$, the partial derivatives exist. Indeed, $\frac{\partial f}{\partial x}(x, 0) = \frac{\partial f}{\partial y}(0, y) = 0$, $\frac{\partial f}{\partial x}(0, y) = \text{sgn}(y)$ and, by symmetry, $\frac{\partial f}{\partial y}(x, 0) = \text{sgn}(x)$, where $\text{sgn}(u)$ is 1 if u is positive, -1 if negative, and 0 if 0. When either $x_0 \neq 0$ or $y_0 \neq 0$, the function is differentiable. For example, for $y_0 > 0$, we have

$$\lim_{(x,y) \rightarrow (0,y_0)} \frac{|\frac{xy}{|x|+|y|} - x|}{(x^2 + (y - y_0)^2)^{1/2}} = \lim_{(x,y) \rightarrow (0,y_0)} \frac{x^2}{(|x| + |y|)(x^2 + (y - y_0)^2)^{1/2}} = 0.$$

However f is NOT differentiable at the origin, because

$$\lim_{(x,y) \rightarrow (0,0)} \frac{|xy|}{(|x| + |y|)(x^2 + y^2)^{1/2}} \neq 0.$$

PS. You can download the graph of the function from the course web-page. By the way, when you look at the graph, you can usually get a very good idea about continuity and differentiability.

5. Solve problem 5 on page 212.

Solution. This is a direct application of the Lemma on page 196.

Extra problems.

9* This one is rather straightforward but long (unless you can come up with something clever). It tells you about the behaviour of two differential operators with respect to a simple change of variables.

Let $u = f(x, y, z)$ be a twice differentiable real-valued function of 3 variables. Let ℓ_1, ℓ_2, ℓ_3 be mutually orthogonal norm-1 vectors. Show that

$$(i) \left(\frac{\partial u}{\partial \ell_1}\right)^2 + \left(\frac{\partial u}{\partial \ell_2}\right)^2 + \left(\frac{\partial u}{\partial \ell_3}\right)^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2.$$

Comment. The norm of the gradient (max rate of change) does not depend on the coordinate system.

$$(ii) \frac{\partial^2 u}{\partial \ell_1^2} + \frac{\partial^2 u}{\partial \ell_2^2} + \frac{\partial^2 u}{\partial \ell_3^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Solution. Let $\ell_j = (\cos \alpha_j, \cos \beta_j, \cos \gamma_j)$, $j = 1, 2, 3$. Since ℓ_1, ℓ_2, ℓ_3 are mutually orthogonal norm-1 vectors, the matrix

$$L = \begin{pmatrix} \cos \alpha_1 & \cos \beta_1 & \cos \gamma_1 \\ \cos \alpha_2 & \cos \beta_2 & \cos \gamma_2 \\ \cos \alpha_3 & \cos \beta_3 & \cos \gamma_3 \end{pmatrix}$$

is unitary (rows and columns form orthonormal bases in E^3). Direct computation shows

$$\begin{aligned} \frac{\partial^2 u}{\partial \ell_j^2} &= \frac{\partial^2 u}{\partial x^2} \cos^2 \alpha_j + \frac{\partial^2 u}{\partial y^2} \cos^2 \beta_j + \frac{\partial^2 u}{\partial z^2} \cos^2 \gamma_j + \\ &+ 2 \frac{\partial^2 u}{\partial x \partial y} \cos \alpha_j \cos \beta_j + 2 \frac{\partial^2 u}{\partial x \partial z} \cos \alpha_j \cos \gamma_j + 2 \frac{\partial^2 u}{\partial y \partial z} \cos \beta_j \cos \gamma_j. \end{aligned} \tag{0.1}$$

Summing over j and using the unitarity of L yields the result.

Homework 9

Problems for all.

Solve problems 8, 9(a,b), 11, 12, and 14 on pages 109–110. As part of problem 12, solve the extra problem from the previous homework (for your convenience, I left it below).

Solutions.

Problem 8.

Applying Rolle's theorem to F we get

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Problem 9.

(a) WLOG a is the left extremity of U . Extend f, g by continuity to a : $f(a) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = g(a) = 0$. Then previous problem implies

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{c \rightarrow a^+} \frac{f'(c)}{g'(c)}.$$

(b) Apply (a) to $1/f$ and $1/g$:

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f^2(x)}{g^2(x)} \cdot \lim_{x \rightarrow a^+} \frac{1/f(x)}{1/g(x)} = \lim_{x \rightarrow a^+} \frac{f^2(x)}{g^2(x)} \cdot \lim_{x \rightarrow a^+} \frac{(1/f)'(x)}{(1/g)'(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}.$$

Problem 11.

By Corollary 3 on p. 105, f' is decreasing (increasing) on U . Therefore, $f' < 0$ ($f' > 0$) to the left of x_0 and $f' > 0$ ($f' < 0$) to the right of x_0 . Apply problem 4 to reach the conclusion.

Problems 12 and 8*. Suppose a function $f : U \rightarrow \mathbb{R}$ is twice differentiable on an open interval U and $[a, b] \subset U$.

(i) Show that if f'' is non-zero everywhere on (a, b) then it is either always positive or negative.

Solution. By the Intermediate value theorem, f' is 1-1. By the version of the inverse function theorem proved in class, f' is strictly monotonic on $[a, b]$. By problem 4, either $f'' > 0$ or $f'' < 0$.

(ii) Show that if $f'' \geq 0$ then for each $c \in [a, b]$ the graph of f lies below the chord line that joins $(a, f(a))$ with $(b, f(b))$ and above the tangent line $y = f(c) + f'(c)(x - c)$.

Solution. We need to show that for all $x, c \in (a, b)$

$$f(c) + f'(c)(x - c) \leq f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) = f(b) + \frac{f(b) - f(a)}{b - a}(x - b). \quad (*)$$

By Mean value theorem, $f(x) - f(c) = f'(d)(x - c)$ for some d between x and c . Since f' is increasing, we have $f(x) - f(c) = f'(d)(x - c) \geq f'(c)(x - c)$. This proves the first inequality.

To prove the second inequality assume that $c \in (a, b)$ is such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Then, for $x \leq c$, we have $f(x) - f(a) = f'(h_a)(x - a) \leq \frac{f(b)-f(a)}{b-a}(x - a)$, since f' is increasing. For $x \geq c$, we have $f(x) - f(b) = f'(h_b)(x - b) \leq \frac{f(b)-f(a)}{b-a}(x - b)$ for the same reason.

(iii) Show that if f is convex then f'' is nonnegative.

Solution. If f'' were known to be continuous you could easily argue by contradiction. Indeed, you could show then by (ii) that f would be concave in the interval where $f'' < 0$. In general, assume that $c \in U$ and $x_n \rightarrow c$ as $n \rightarrow \infty$ with $x_n > c$. Let d_n be such that $f(x_n) - f(c) = f'(d_n)(x_n - c)$ and $d_n \in (c, x_n)$. By (*), we have $f'(d_n) \geq f'(c)$. However, since $f''(c)$ exists,

$$f''(c) = \lim_{n \rightarrow \infty} \frac{f'(d_n) - f'(c)}{d_n - c} \geq 0.$$

(iv) Assuming only that f is ONCE differentiable on U show that one inequality in (*) implies the other, *i.e.*

$$f(c) + f'(c)(x - c) \leq f(x) \quad \forall (a, b) \subset U \text{ and } \forall x, c \in (a, b)$$

is equivalent to

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) = f(b) + \frac{f(b) - f(a)}{b - a}(x - b) \quad \forall (a, b) \subset U \text{ and } \forall x \in (a, b).$$

Solution. \Rightarrow . Assume for contradiction that there exist $(a, b) \subset U$ and $x \in (a, b)$ such that

$$(1) \quad f(x) > f(a) + \frac{f(b) - f(a)}{b - a}(x - a) = f(b) + \frac{f(b) - f(a)}{b - a}(x - b)$$

but

$$(2) \quad f(x) + f'(x)(a - x) \leq f(a)$$

and

$$(3) \quad f(x) + f'(x)(b - x) \leq f(b).$$

From (1) and (2), $\frac{f(b)-f(a)}{b-a}(x - a) < f(x) - f(a) \leq f'(x)(x - a)$ which implies $\frac{f(b)-f(a)}{b-a} < f'(x)$ since $x > a$. However, from (1) and (3), $\frac{f(b)-f(a)}{b-a}(x - b) < f(x) - f(b) \leq f'(x)(x - b)$ which implies $\frac{f(b)-f(a)}{b-a} > f'(x)$ since $x < b$. This is a contradiction.

\Leftarrow . Consider the “chord inequality” with $x = a + \lambda(b - a)$. We get

$$f(a + \lambda(b - a)) \leq f(a) + \frac{f(b) - f(a)}{b - a}(a + \lambda(b - a) - a) = f(a) + \lambda(f(b) - f(a)).$$

This implies

$$\frac{f(a + \lambda(b - a)) - f(a)}{\lambda(b - a)}(b - a) \leq f(b) - f(a).$$

Taking the limit as $\lambda \rightarrow 0^+$, we get the “tangent line inequality.”

Problem 14.

Let $f(x) = (a + x)^n$. Then $f^{(k)}(x) = \frac{n!}{(n-k)!}(a + x)^{n-k}$ for $k \leq n$ and $f^{(k)}(x) = 0$ for $k > n$. Hence, by Taylor’s theorem,

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + 0 = \sum_{k=0}^n \frac{n!}{k!(n-k)!} a^{n-k} x^k.$$

Homework 8

Problems for all.

Solve problems 1–4 and 6 on pages 108–109.

Solutions.

Problem 1.

(a) This function is clearly differentiable on $\mathbb{R} \setminus \{0\}$. However, since $\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} \sin \frac{1}{x}$ does not exist, f is not differentiable at 0.

(b) This function is differentiable. At 0 we have $\lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$. Observe, however, that on $\mathbb{R} \setminus \{0\}$ we have $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Hence f' is not continuous.

(c) On $\mathbb{R} \setminus \{0\}$ we have $f'(x) = \frac{1}{2\sqrt{|x|}}$. At 0, however, f is not differentiable: $\lim_{x \rightarrow 0} \frac{|f(x)|}{|x|} = \infty$. Observe that the tangent line at 0 exists.

Problem 2.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x_0 + \alpha h) - f(x_0 + \beta h)}{h} &= \lim_{h \rightarrow 0} \frac{f(x_0 + \alpha h) - f(x_0) + f(x_0) - f(x_0 + \beta h)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{f(x_0 + \alpha h) - f(x_0)}{h} + \lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 + \beta h)}{h} = \\ &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{\alpha^{-1}h} - \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{\beta^{-1}h} = (\alpha - \beta)f'(x_0) \end{aligned}$$

for $\alpha, \beta \in \mathbb{R} \setminus \{0\}$. With a little justification the formula holds for all $\alpha, \beta \in \mathbb{R}$.

Problem 3.

The problem with the “proof” is best seen when $f = \text{const}$. Then $\lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}$ doesn't make sense. Observe that $g'(f(x_0)) = \lim_{y \rightarrow f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)}$. The correct proof is on page 103.

Problem 4.

One direction is proved exactly as corollary 4 on page 105. The other one follows immediately from the definition of the derivative and the properties of the limit.

Problem 6.

Let $M = \sup f'$. Then $|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|$ implies uniform continuity.

Extra problems.

8* This should've been a regular problem but I haven't prepared you for it yet. We will probably get back to it later.

Suppose a function $f : [a, b] \rightarrow \mathbb{R}$ is differentiable on $[a, b]$ and has a non-zero second derivative everywhere on (a, b) .

(i) Show that f'' is either always positive or negative.

(ii) Show that if f has the above property then for each $c \in [a, b]$ the graph of f lies between the chord line that joins $(a, f(a))$ with $(b, f(b))$ and the tangent line $y = f(c) + f'(c)(x - c)$.

Remark. This problem is relatively easy to solve when f'' is continuous. Here you need to do it without the continuity assumption.

Homework 7

Problems for all.

1. Solve problem 32 on page 94.

Solution. We write the sequence in the form $f_0(x) = \sqrt{x}$, $f_n(x) = \sqrt{f_{n-1}(x) + x}$. Since $f_1(x) > f_0(x)$, we have $f_n(x) > f_{n-1}(x)$ for all $n \in \mathbb{N}$. Thus, $\{f_n(x)\}_{n \in \mathbb{N}}$ is an increasing sequence for all $x > 0$. Next, we prove that $f_n(x) < x + 1$ for all $n \in \mathbb{N}$. This is clearly true for f_0 and for $n > 0$ we have $f_n(x) = \sqrt{f_{n-1}(x) + x} < \sqrt{x + 1 + x} < \sqrt{x^2 + 2x + 1} = x + 1$ by induction. Hence, the sequence $\{f_n(x)\}_{n \in \mathbb{N}}$ is bounded above and the monotone convergence theorem asserts existence of the limit $L(x)$. The latter must satisfy $L(x) = \sqrt{L(x) + x}$, which yields $L(x) = \frac{1 + \sqrt{1 + 4x}}{2}$.

2. Solve problem 35 on page 94.

Solution. By definition of uniform continuity, for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ as soon as $|x - y| < \delta$. Hence, for every $\epsilon > 0$ and $x \in \mathbb{R}$ there exists $N > \frac{1}{\delta}$ such that for every $n > N$ we have $|f(x + \frac{1}{n}) - f(x)| < \epsilon$. Therefore, $\{f(x + \frac{1}{n})\}$ converges to f uniformly.

3. Solve problem 39 on page 94.

Solution. The example provided below is a modification of the sequence $f_n(x) = x^n$ that converges to a function discontinuous at 1. For $n \in \mathbb{N}$, let

$$g_n(x) = \begin{cases} 2^{kn}(x - 2^{-k})^n, & x \in (2^{-k}, 2^{1-k}], \quad k \in 2\mathbb{N} - 1; \\ 2^{kn}(2^{1-k} - x)^n, & x \in (2^{-k}, 2^{1-k}], \quad k \in 2\mathbb{N}; \\ 0, & x = 0. \end{cases}$$

It is not hard to check that these functions satisfy all the desired properties but one – they are not continuous at 0. To remedy this, we let $f_n(x) = xg_n(x)$.

4. Let S be a bounded subset of \mathbb{R}^n and $f : S \rightarrow \mathbb{R}$ a continuous function. Show that f extends to a continuous function \bar{f} from the closure \bar{S} into \mathbb{R} iff f is uniformly continuous on S .

Solution. The “only if” part is trivial because any continuous function on a compact set \bar{S} (and, hence, on S) is uniformly continuous. The ‘if’ part is a little tricky. First of all, for $b \in \bar{S} \setminus S$ we define $\bar{f}(b) = \lim_{n \rightarrow \infty} f(x_n)$ for some sequence $x_n \in S$ that converges to b . We can do this because the limit exists and does not depend on the choice of the sequence. Indeed, it exists because $f(x_n)$ is Cauchy in \mathbb{R} (that is an easy consequence of the uniform continuity which I leave for you). Also, if y_m is another sequence that converges to b then $|\bar{f}(b) - f(y_m)| \leq |\bar{f}(b) - f(x_n)| + |f(y_m) - f(x_n)|$. Since f is uniformly continuous on S , for every $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$, $x, y \in S$, implies $|f(x) - f(y)| < \epsilon/2$. But since $x_n \rightarrow b$ and $y_m \rightarrow b$, for every $\delta > 0$ we can pick $N \in \mathbb{N}$ such that $|y_m - x_n| \leq |y_m - b| + |b - x_n| < \delta$ for all $m, n > N$. Hence, for all $n, m > N$, $|f(y_m) - f(x_n)| < \epsilon/2$. Clearly, $|\bar{f}(b) - f(x_n)|$ is less than $\epsilon/2$ for big n because $\bar{f}(b) = \lim_{n \rightarrow \infty} f(x_n)$. Thus, \bar{f} is well-defined on \bar{S} . Now we need to show that \bar{f} is continuous on \bar{S} . We do this by means of the proposition on page 74 of your book. Let $p_n \in \bar{S}$ be an arbitrary sequence that

converges to $p \in \bar{S}$. We need to show that for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|f(p_n) - f(p)| < \epsilon$ for all $n > N$. Let $p_n^m \in S$ be such that $p_n = \lim_{m \rightarrow \infty} p_n^m$. Clearly, we can always choose a “subsequence” $p_n^{m(n)}$ that converges to p as $n \rightarrow \infty$, so that $|f(p_n) - f(p)| \leq |f(p_n) - f(p_n^{m(n)})| + |f(p_n^{m(n)}) - f(p)| < \epsilon$.

5. Solve problem 42 on page 95.

Solution. The unit ball in $\mathcal{C}[0, 1]$ is the set $\bar{B}(0, 1) = \{f \in C[0, 1] : |f(x)| \leq 1\}$. Clearly, The functions $f_n(x) = x^n$ belong to this set. If the set were compact than this sequence would have a convergent subsequence. But $\lim_{n \rightarrow \infty} f_n(x) = \chi_{\{1\}}(x)$ is not continuous and, therefore, not in $\bar{B}(0, 1)$. Since the limit is unique, a uniform limit is also a pointwise limit, and any subsequence of a convergent sequence converges to the same limit, we have that no subsequence of this sequence converges in $\mathcal{C}[0, 1]$. This is a contradiction.

Extra problems.

7* This exercise leads to another proof of the Weierstrass approximation theorem. This proof is kind of “off-the-wall” in a sense that it’s easy to understand but it’s completely unintuitive and not instructive. Since I don’t want you to hate analysis, I did not present it in class. On the other hand it maybe useful to know that it exists and it might also be a good weapon in the fight against your laziness. :)

Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. For every $n \in \mathbb{N}$ define

$$\tilde{f}_n(t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} f\left(\frac{k}{n}\right).$$

Show that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall n > N, D(\tilde{f}_n, f) < \epsilon$.

Hints: Show that

$$\tilde{f}_n(t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} = 1;$$

$$\tilde{f}_n(t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} (k - nt) = 0;$$

$$\tilde{f}_n(t) = \sum_{k=0}^n \binom{n}{k} t^k (1-t)^{n-k} \left(t - \frac{n}{k}\right)^2 = \frac{t(1-t)}{n}.$$

Use the above relations to estimate $|f(t) - \tilde{f}_n(t)|$. You will need to consider two sums separately:

$$\sum_{k: |t - \frac{n}{k}| < \delta} \quad \text{and} \quad \sum_{k: |t - \frac{n}{k}| \geq \delta} .$$

Solution. See <http://www.math.toronto.edu/~akricker/weierstrass.pdf>

Homework 6

Problems for all.

1. Solve problem 10 on page 92.

Solution.

- (a) f is continuous at all points other than the origin as a rational function. f is not continuous at $(0, 0)$ because $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist. Indeed, for any $M > 0$ there exist $x = y = \frac{1}{2\sqrt{M}}$ so that $f(x, y) \geq M$.
- (b) f is continuous at all points other than the origin as a rational function. f is not continuous at $(0, 0)$ because $\lim_{x \rightarrow 0} f(x, x) = \frac{1}{2} \neq 0$. Indeed, if $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ existed we would have $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \lim_{x \rightarrow 0} f(x, x)$ since $B((0, 0), \delta) \supset \{(x, x) : x \in B(0, \frac{\delta}{\sqrt{2}})\}$.
- (c) f is continuous because $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$. Indeed, $y^2 \leq x^2 + y^2$ implies $|f(x, y)| \leq |x|$.

2. Solve problem 14 on page 92.

Solution.

- (a) Define $f : S \rightarrow \mathbb{R}$ by $f(p) = d(p, p_0)$. By Example 2 on page 62, f is continuous. Since S is compact, the minimum exists by Corollary 2 on page 78.
- (b) Let $M > 0$ be such that $\bar{B}(p_0, M) \cap S \neq \emptyset$. Replacing S with $\bar{B}(p_0, M) \cap S$, we reduce the problem to the previous one.

3. Solve problem 19 on page 92.

Solution. Define $f : E \rightarrow \mathbb{R}$ by $f(p) = d(p, p_0)$. Then $|f(p) - f(q)| = |d(p, p_0) - d(q, p_0)| \leq d(p, q)$ implies uniform continuity.

4. In this problem you are to prove one of the theorems of Cauchy, which is also a continuous analogue of Problem 2 in HW 4.

Recall that $\lim_{x \rightarrow +\infty} f(x) = L$ if for every $\epsilon > 0$ there exists $X \in \mathbb{R}$ such that $x > X$ implies $|f(x) - L| < \epsilon$. Assume that $f : [0, \infty) \rightarrow [C, \infty)$ for some $C > 0$ is bounded on bounded intervals and $\lim_{x \rightarrow +\infty} \frac{f(x+1)}{f(x)} = M < \infty$. Show that $\lim_{x \rightarrow +\infty} [f(x)]^{\frac{1}{x}} = M$.

Solution. There is a strong temptation to reduce this problem to the earlier one about sequences. Indeed, let $a_n(x) = f(x+n)$. Clearly, for every $x \in (0, 1]$ we have $\lim_{n \rightarrow \infty} \frac{a_{n+1}(x)}{a_n(x)} = M$. By Problem 2 in HW 4, we have $\lim_{n \rightarrow \infty} [a_n(x)]^{\frac{1}{n}} = M$ for every $x \in (0, 1]$. Hence, $\lim_{n \rightarrow \infty} [f(n+x)]^{\frac{1}{n+x}} = \lim_{n \rightarrow \infty} \left([a_n(x)]^{\frac{1}{n}} \right)^{\frac{n}{n+x}} = M$ for every $x \in (0, 1]$.

Now we would like to conclude that $\lim_{x \rightarrow +\infty} [f(x)]^{\frac{1}{x}} = M$. However, we can not do that, unless we show that all the sequence limits mentioned above converge **uniformly** for $x \in (0, 1]$. This, in fact, is true, but is tantamount to retracing the solution of Problem 2 in HW 4. Thus, it may be easier to solve the problem directly.

Let's denote by $[x]$ the biggest integer that is less than or equal to x and set $\{x\} = x - [x]$. Since $\lim_{x \rightarrow +\infty} \frac{[x]}{x} = 1$ it is enough to show that $\lim_{x \rightarrow +\infty} [f(x)]^{\frac{1}{[x]}} = M$. Pick $\epsilon > 0$. Then there exists $N_\epsilon \in \mathbb{N}$ such that $M - \epsilon < \frac{f(x+1)}{f(x)} < M + \epsilon$ for all $x \geq N_\epsilon$. Now

$$(M - \epsilon)^{[x] - N_\epsilon} < \frac{f(x)}{f(N_\epsilon + \{x\})} = \frac{f(x)}{f(x-1)} \times \dots \times \frac{f(f(N_\epsilon + 1 + \{x\}))}{f(N_\epsilon + \{x\})} < (M + \epsilon)^{[x] - N_\epsilon}.$$

Since f is bounded on bounded intervals, $f(N_\epsilon + \{x\}) < B_\epsilon$ for some $B_\epsilon \in [C, \infty)$. Hence,

$$\frac{C}{(M - \epsilon)^{N_\epsilon}} (M - \epsilon)^{[x]} < f(x) < \frac{B_\epsilon}{(M + \epsilon)^{N_\epsilon}} (M + \epsilon)^{[x]} \text{ and}$$

$$\left(\frac{C}{(M - \epsilon)^{N_\epsilon}} \right)^{\frac{1}{[x]}} (M - \epsilon) < [f(x)]^{\frac{1}{[x]}} < \left(\frac{B_\epsilon}{(M + \epsilon)^{N_\epsilon}} \right)^{\frac{1}{[x]}} (M + \epsilon).$$

Since $\lim_{x \rightarrow +\infty} \left(\frac{D}{(M + \epsilon)^{N_\epsilon}} \right)^{\frac{1}{[x]}} = 1$ for any constant D , we have that $M - 2\epsilon < [f(x)]^{\frac{1}{[x]}} < M + 2\epsilon$ for all x greater than some $R \in \mathbb{R}$.

5. Solve problem 31 on page 94.

Solution.

- (a) Suppose $r = .r_1 r_2 r_3 r_4 \dots \notin S$. Then $r_{4k} \neq 0$ for some $k \in \mathbb{N}$. This implies that $\text{dist}(r, S) > 10^{-4k-5}$. Hence, $r \notin \bar{S}$. This implies that $S = \bar{S}$.
- (b) Continuity follows because $|s - q| < 10^{-4k}$ implies $|\varphi_i(s) - \varphi_i(q)| < 10^{-k}$ for $i = 1, 2, 3$.
- (c) Since S is closed, S^c is open. As we mentioned before, S^c is a disjoint union of bounded intervals, which we call $I_j = (l_j, r_j)$, $j \in \mathbb{N}$. Clearly, $l_j, r_j \in S$ for all $j \in \mathbb{N}$. For $s \in I_j$ we can define

$$f_i(s) = \frac{r_j - s}{r_j - l_j} \varphi_i(l_j) + \frac{s - l_j}{r_j - l_j} \varphi_i(r_j).$$

Clearly, there are no other functions that satisfy stated properties. These functions are continuous because they are monotonic on each I_j and φ_i 's are continuous.

- (d) f is continuous because each f_i is continuous. f is onto $[0, 1]^3$ because

$$f(.a_1 b_1 c_1 0 a_2 b_2 c_2 0 a_3 b_3 c_3 \dots) = (.a_1 a_2 a_3 \dots, .b_1 b_2 b_3 \dots, .c_1 c_2 c_3 \dots).$$

Extra problems.

6* Find a monotonic function $f : [0, 1] \rightarrow \mathbb{R}$ which is continuous only at irrational points.

Solution. Let $\mathbb{Q} = \{r_n, n \in \mathbb{N}\}$ and $f = \chi_{\mathbb{R}_+}$. Set

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n).$$

Homework 5

Problems for all.

1. Solve problem 1 on page 90.

Solution.

- (a) f is continuous. It is continuous on $(0, +\infty)$ as the identity function and on $(-\infty, 0)$ as a constant function. It is continuous at 0 because $|f(x) - f(0)| = |f(x)| \leq |x| = |x - 0|$.
- (b) f is continuous because $\lim_{x \rightarrow 0} f(x) = 0$, since $\sin \frac{1}{x}$ is bounded. Indeed, $|f(x) - f(0)| \leq |x| = |x - 0|$.
- (c) f is not continuous at 0, because $\lim_{x \rightarrow 0} f(x) = 0 \neq 1$. It is continuous at all other points as shown in Example 1 on p. 69.
- (d) Clearly, f is not continuous at any rational point. This follows from the fact that any interval in \mathbb{R} contains irrational points and $0 \notin f(\mathbb{Q})$. Surprisingly, f is continuous at all irrational points. Indeed, if $x \notin \mathbb{Q}$ and $q \in \mathbb{N}$, the set $S_q = \{|x - \frac{p}{q}|, p \in \mathbb{Z}\}$ is closed and bounded below. Hence, there exists $p \in \mathbb{Z}$ such that $\inf S_q = |x - \frac{p}{q}| > 0$. Clearly, $y \in B(x, \inf S_q)$ implies $|f(y)| < \frac{1}{q}$. Thus, f is continuous at x .

2. Solve problem 2 on page 91.

Solution. If $S \subseteq E'$ is closed, S^c is open. Since f is continuous, $f^{-1}(S^c) = (f^{-1}(S))^c$ is open. Hence $f^{-1}(S)$ is closed. The rest follows because \mathbb{R}_+ , \mathbb{R}_- , and $\{0\}$ are closed.

3. Solve problem 3 on page 91.

Solution. Let M be a closed subset of E' . Let us denote by f_1 and f_2 the restrictions of f on S_1 and S_2 respectively. Then $f^{-1}(M) = (f^{-1}(M) \cap S_1) \cup (f^{-1}(M) \cap S_2) = f_1^{-1}(M) \cup f_2^{-1}(M)$ is closed as a union of two closed sets.

4. In this problem you are supposed to provide different examples. You are not allowed to use the ones in the text.

- (a) Show that if $f : E \rightarrow E$ is a constant function, then it is continuous.
- (b) Find a continuous function $f : (0, 1) \cup (2, 3) \rightarrow \mathbb{R}$ that is not constant.
- (c) Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is discontinuous at every point.
- (d) Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous only at two points.
- (e) Find a continuous function $f : E \rightarrow E$ such that $f(E) = \mathbb{Q}$.

Solution.

- (a) Let $S \subseteq E$ be an open set and $f(x) \equiv c$. If $c \in S$ then $f^{-1}(S) = E$ is open. If $c \notin S$ then $f^{-1}(S) = \emptyset$ is open.
- (b) $f : (0, 1) \cup (2, 3) \rightarrow \mathbb{R}$, $f(x) = 2x$ is continuous (either as a product of 2 continuous functions or directly from the definition: $|2x - 2p| = 2|x - p| < \delta = \epsilon$).

- (c) Let $S \in \mathbb{R}$ be such that $S^o = (S^c)^o = \emptyset$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a characteristic (indicator) function of S :

$$f(x) = \chi_S(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

By construction, for any point in S there is a sequence of points in S^c , which converges to S and vice versa.

- (d) Let $\chi_{\mathbb{Q}}$ be as above. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = x(1-x)\chi_{\mathbb{Q}}$. Clearly, $|f(x) - f(1)| = |f(x) - f(0)| = |x-0||1-x|\chi_{\mathbb{Q}} \leq |x-0||1-x| < C\delta$. Hence, f is continuous at 0 and 1. It is easy to show that it is discontinuous at all other points.
- (e) $f : \mathbb{Q} \rightarrow \mathbb{Q}$, $f(x) = 2x$ (you are not allowed to use the identity).

5. Solve problem 5 on page 91.

Solution. Let f be continuous at p then for any $\epsilon > 0$ there exists $\delta > 0$ such that we have $d'(f(x), f(y)) \leq d'(f(x), f(p)) + d'(f(p), f(y)) < \epsilon$ as soon as $x, y \in B(p, \delta)$. Hence, the oscillation is 0. Conversely, if the oscillation of f at p is 0 ($osc_f(p) = 0$), the definition of the inf implies that for any $\epsilon > 0$ there exists $\delta > 0$ such that $d'(f(p), f(x)) < \epsilon$ as soon as $x \in B(p, \delta)$ (this is true because $p \in B(p, \delta)$ for all $\delta > 0$). Hence, f is continuous at p .

Let $S(\epsilon) = \{p \in E : osc_f(p) < \epsilon\}$. We want to show that $S(\epsilon)$ is open for every $\epsilon > 0$. If $osc_f(p) < \epsilon$, then there exists $\delta > 0$ such that $x, y \in B(p, \delta)$ implies $d'(f(x), f(y)) < \epsilon$. But $s \in B(p, \delta/2)$ implies $B(s, \delta/2) \subseteq B(p, \delta)$. Hence, $osc_f(s) < \epsilon$ for all $s \in B(p, \delta/2)$, q.e.d.

6. Solve problem 6 on page 91. Can such a function f be continuous if $S \neq E$?

Solution. Let $x \in \partial S$. WLOG, $x \in S$. Since $x \in \partial S$ there exists a sequence $\{x_n\} \subseteq S^c$ which converges to x . Clearly, $\lim f(x_n) = 0 \neq 1 = f(x)$ and f is discontinuous. Conversely, let $x \notin \partial S$. WLOG, $x \in S^o$. Then for any sequence x_n is E that converges to x $\lim f(x_n) = 1 = f(x)$ and f is continuous. Clearly, such a function is continuous iff $\partial S = \emptyset$. This is true whenever S is clopen. Thus, if E is not connected S can be a proper subset.

Extra problems.

- 6* Find a function $f : [0, 1] \rightarrow \mathbb{R}$ which is continuous only at rational points.

Solution. I have to apologize for this one. What I meant to ask was to find a **monotonic** function $f : [0, 1] \rightarrow \mathbb{R}$ which is continuous only at **irrational** points. The one I actually asked for **does not exist**. The proof is well beyond the scope of this course but, I believe, some of you can understand it. So, here it is.

Proof. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is continuous at all rational points. Consider the collection of sets $S(\epsilon)$ defined in the solution of problem 5 and let $U_n = S(\frac{1}{n}) \setminus \{r_n\}$, $n \in \mathbb{N}$, where $\{r_n, n \in \mathbb{N}\} = \mathbb{Q} \cap [0, 1]$. Clearly, $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} U_n \setminus \mathbb{Q} \subset \{x \in [0, 1] : osc_f(x) = 0\} = \{x \in [0, 1] : f \text{ is continuous at } x\}$. But, as follows from problem 5, U_n is open for every $n \in \mathbb{N}$ and it is also *dense*, i.e., $\overline{U_n} = [0, 1]$, because $\overline{\mathbb{Q}} = \mathbb{R}$. Hence, we

can apply Baire's category theorem which says that $\bigcap_{n=1}^{\infty} U_n$ is dense in $[0, 1]$. Hence, f is continuous not only at rational points but also on a dense set of irrational points.

Homework 4

Problems for all.

1. Let (E, d) be a metric space and $S \subset E$. The interior S^o , closure \bar{S} , and boundary ∂S of the set S are defined by

$$S^o = \{x \in S : \text{there exists } r > 0 \text{ such that } B(x, r) \subset S\};$$

$$\partial S = \{x \in E : (\forall r > 0)((B(x, r) \cap S \neq \emptyset) \text{ and } (B(x, r) \setminus S \neq \emptyset))\};$$

$$\bar{S} = S \cup \partial S.$$

Recall that $d(x, S) = \inf\{d(x, p) : p \in S\}$. By S^c we will denote the complement of S . Prove that

- (i) S^o is open and $S^o = \{x \in S : d(x, S^c) > 0\}$.
- (ii) \bar{S} is closed and $\bar{S} = \{x \in E : d(x, S) = 0\} = ((S^c)^o)^c$.
- (iii) ∂S is closed and $\partial S = \partial S^c = \{x \in E : d(x, S) = d(x, S^c) = 0\}$.

Define the same three sets in the “language of sequences”.

Solution.

(i) Openness of S^o is immediate from the definition. The rest follows from the facts that $B(x, d(x, S^c)) \subseteq S$ and $B(x, d(x, S^c) + \epsilon) \setminus S \neq \emptyset$ for all $\epsilon > 0$.

(ii, iii) $d(x, S) = d(x, S^c) = 0$ is equivalent to $(\forall r > 0)((B(x, r) \cap S \neq \emptyset) \text{ and } (B(x, r) \cap S^c \neq \emptyset))$ and, hence, to $x \in \partial S$. Since the definition of ∂S is symmetric w.r.t. S and S^c , $\partial S = \partial S^c$. Now it is obvious that $\bar{S} = \{x \in E : d(x, S) = 0\}$. Furthermore, $(x \in ((S^c)^o)^c) \Leftrightarrow (x \notin (S^c)^o) \Leftrightarrow (d(x, S) = 0) \Leftrightarrow (x \in \bar{S})$. Now, \bar{S} is closed as a complement of an open set and $\partial S = \bar{S} \cap \bar{S}^c$ is closed as an intersection of 2 closed sets.

2. Problem 18 on page 63 provides definitions of $\limsup_{n \rightarrow \infty} a_n$ and $\liminf_{n \rightarrow \infty} a_n$. Two important criteria of convergence of the series involve the limits of $\frac{a_{n+1}}{a_n}$ and $a_n^{1/n}$.

- (i) Show that

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} a_n^{1/n} \leq \limsup_{n \rightarrow \infty} a_n^{1/n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}.$$

- (ii) Give an example when $\lim_{n \rightarrow \infty} a_n^{1/n}$ exists but $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ does not, although, by (i) existence of the latter implies existence of the former.

Solution. (i) We do only the “lim sup” part, “lim inf” part follows by taking $b_n = a_n^{-1}$. Suppose that $R = \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < \infty$, otherwise there is nothing to prove. Pick $\epsilon > 0$. By definition of the “lim sup”, there exists n_ϵ for which $\frac{a_{n+1}}{a_n} < R + \epsilon$ for all $n \geq n_\epsilon$. Hence, for $n > n_\epsilon$,

$$\frac{a_n}{a_{n_\epsilon}} = \frac{a_{n_\epsilon+1}}{a_{n_\epsilon}} \cdot \frac{a_{n_\epsilon+2}}{a_{n_\epsilon+1}} \cdots \frac{a_n}{a_{n-1}} < (R+\epsilon)^{n-n_\epsilon} \Rightarrow a_n^{1/n} < \left(\frac{a_{n_\epsilon}}{(R+\epsilon)^{n_\epsilon}} \right)^{\frac{1}{n}} (R+\epsilon) = C(\epsilon)^{1/n} (R+\epsilon).$$

Since $\lim_{n \rightarrow \infty} C(\epsilon)^{1/n} = 1$, we can conclude that $\limsup_{n \rightarrow \infty} a_n^{1/n} < R + 2\epsilon$.

(ii) Take $a_{2n} = 2$ and $a_{2n-1} = 1$. Then $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1/2$, $\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2$, but $\lim_{n \rightarrow \infty} a_n^{1/n} = 1$.

3. Solve problem 26 on page 64.

Solution. Any point in $S = \{0\} \cup \{1/n, n \in \mathbb{N}\}$ clearly is a cluster point. We need to show that there are no other cluster points. Points outside $[0, 1]$ are obviously not cluster points. Let x be an arbitrary point in $[0, 1] \setminus S$. Then there exists $k \in \mathbb{N}$ and $\epsilon > 0$ such that $\frac{1}{k+1} + \epsilon < x < \frac{1}{k} - \epsilon$, i.e. $x \in A(k, \epsilon) = (\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon)$. Assume, now, that $\frac{1}{n} + \frac{1}{m} \in A(k, \epsilon)$. WLOG, $m > n$. This implies $n < 2(k+1)$ and $\frac{1}{m} > \epsilon$. Clearly, there are at most finitely many such n and m . Thus, x is not a cluster point.

4. Solve problem 30 on page 64.

Solution. (a) \mathbb{N} ; (b) closed unit ball in $\ell^\infty(\mathbb{R})$; (c) \mathbb{Q} .

5. Solve problem 32 on page 64.

Solution. Let $K = K_1 \cup K_2 \cup \dots \cup K_n$, where all K_i are compact. Let $\{U_\alpha\}$ be an open cover of K . Then $\{U_\alpha\}$ is an open cover for all K_i . Since each of the K_i is compact, there are n finite subcovers each of which covers the corresponding K_i . The union of these subcovers is a finite subcover of K . Alternatively, let $\{x_n\}$ be a sequence in K . Then there exists i such that K_i contains infinitely many members of $\{x_n\}$. They form a subsequence in K_i . By compactness, it has a convergent subsequence.

6. Solve problem 38 on page 65.

Solution.

Suppose E is a disjoint union of two nonempty open (closed) subsets. Then one of them is the complement of the other, which implies that they are both open and closed (clopen). Since they are both nonempty, E is not connected. Conversely, if E is not connected, let A be a proper clopen subset of E . Then $E = A \cup A^c$ is a disjoint union of two proper clopen subsets.

7. Let S be a connected subset in a metric space. Show that \bar{S} is connected.

Solution. Let A be a clopen subset of \bar{S} . Then $A \cap S$ is a clopen subset in the space S . Hence either $A \cap S = S$ or $A \cap S = \emptyset$ (because S is connected). If $A \cap S = S$, we have $A = \bar{S}$ because A is closed in \bar{S} . In case $A \cap S = \emptyset$, we have $A = \emptyset$ because A is open in \bar{S} . This follows from the fact that $(\partial S)^\circ = \emptyset$, by Problem 1.

Extra problems.

- 5* This is an extension of the nested set property on page 55. Let us call a family $\{S_\alpha\}$ of subsets of a metric space E *centered* if any finite intersection of its members is non-empty. Prove that E is compact if and only if every centered family of its closed subsets has a non-empty intersection.

Solution. Let $\{S_\alpha\}$ be a centered family of closed subsets of a compact metric space E . The sets $U_\alpha = E \setminus S_\alpha$ are open and no finite subfamily of $\{U_\alpha\}$ covers E because $\{S_\alpha\}$ is centered. By compactness, the family $\{U_\alpha\}$ does not cover E . Hence, $\bigcap S_\alpha \neq \emptyset$.

Conversely, assume that any centered family of closed subsets of a metric space E has non-empty intersection. Let $\{U_\alpha\}$ be an open cover of E . Let $S_\alpha = E \setminus U_\alpha$. Then $\bigcap S_\alpha = \emptyset$ and, by assumption, the family S_α is not centered. Hence, There exists a finite subfamily S_1, S_2, \dots, S_n such that $\bigcap_{i=1}^n S_i = \emptyset$. But this implies that $U_i = E \setminus S_i$ form a finite subcover. Hence, E is compact.

Homework 3

Problems for all.

1. Solve problem 1(b) on page 61.

Solution. We need to check 3 properties of the metrics:

(i) $d(x, y) \geq 0$ – obvious. ($d(x, y) = 0$) iff ($x_i = y_i$ for all i) iff ($x = y$).

(ii) $d(x, y) = d(y, x)$ by the properties of the modulus.

(iii) $d(x, y) = \sup\{|x_i - y_i|\} \leq \sup\{|x_i - z_i| + |z_i - y_i|\} \leq \sup\{|x_i - z_i|\} + \sup\{|y_i - z_i|\} = d(x, z) + d(z, y)$.

2. Let (X, d) be a metric space. Define $\rho : X \times X \rightarrow \mathbb{R}_+$ by $\rho(x, y) = \ln(1 + d(x, y))$. Show that (X, ρ) is a metric space.

Solution. Again the first 2 properties of the metrics are obvious. The third one follows from $\ln(1 + d(x, y)) \leq \ln(1 + d(x, z) + d(y, z)) \leq \ln(1 + d(x, z) + d(y, z) + d(x, z)d(y, z)) = \ln((1 + d(x, z))(1 + d(y, z))) = \ln(1 + d(x, z)) + \ln(1 + d(y, z)) = \rho(x, z) + \rho(y, z)$.

3. Give an example of a metric space (X, d) and two balls $B_0 = B(p_0, r_0)$ and $B_1 = B(p_1, r_1)$ such that $r_0 < r_1$ but B_1 is properly contained in B_0 .

Solution. Let d be the usual Euclidean metrics in \mathbb{R}^2 and set $X = B_0 = B(\mathbf{0}, 1)$. Then $B_1 = B((1, 0), \frac{3}{2}) \cap X$ is clearly a ball of bigger radius properly contained in B_0 . Note: in this solution all the balls are closed ones.

4. Solve problem 5 on page 61.

Solution. I'm afraid, you would not be able to provide a completely rigorous prove of this statement without going into countability argument or giving the one which is tantamaount to rediscovering of the basic properties of equivalence relations. You can talk to me if you want to learn any of these. Your grader will evaluate your solutions on an “effort basis”. What you could have observed/proved includes:

(i) Since the set is open, it is equal to the union of intervals contained in it and surrounding each of its points;

(ii) The union of two intersecting intervals contained in a bounded set is itself an interval;

(iii) For any point in a bounded open set S there exists an interval around this point which is contained in S and contains all the other such intervals.

5. Solve problem 11 on page 62.

Solution. First of all, observe that for any fixed $N \in \mathbb{N}$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^N a_i = 0, \text{ and, hence, } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n a_i = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=N+1}^n a_i = \lim_{n \rightarrow \infty} \frac{1}{n-N} \sum_{i=N+1}^n a_i.$$

Now, for an arbitrary $\varepsilon > 0$ pick $N \in \mathbb{N}$ such that $|a_i - a| \leq \varepsilon$ for all $i \geq N$. Then

$$\left| \frac{1}{n-N} \sum_{i=N+1}^n a_i - a \right| \leq \frac{1}{n-N} \sum_{i=N+1}^n |(a_i - a)| < \varepsilon.$$

6. Solve problem 13 on page 62.

Solution. The sequence is defined recursively by $a_{n+1} = \frac{1}{2+a_n}$ with $a_0 = \frac{1}{2}$ and $a_1 = \frac{2}{5}$. Hence, if the limit L exists, it would be positive and satisfy $L = \frac{1}{2+L}$, i.e., $L = \sqrt{2} - 1$. To show existence of the limit, we use the hint. Indeed, the function $f(x) = \frac{1}{2+\frac{1}{2+x}}$ is obviously increasing on \mathbb{R}_+ and L is the only positive fixed point of this function. Since $a_0 > L$ the subsequence of odd terms is decreasing and since $a_1 < L$ the subsequence of even terms is increasing. Since each of them is bounded below by 0 and above by 1, the monotone convergence theorem applies. Now you have to fuss a little bit to show that these subsequences have the same limit and that this implies existence of the limit of the original sequence. Let $g(x) = \frac{1}{2+x}$. Since, L is the fixed point of g it is also the fixed point of $f = g \circ g$. This implies $\lim a_{2n} = \lim a_{2n-1} = L$. The rest follows immediately from the definition of the limit.

7. Here is another limit, a little more difficult. Find $\lim_{n \rightarrow \infty} \sin^2(\pi\sqrt{n^2+n})$ or show that it does not exist.

Solution. Using an easy trig identity (and continuity) we get $\lim_{n \rightarrow \infty} \sin^2(\pi\sqrt{n^2+n}) = \lim_{n \rightarrow \infty} \sin^2(\pi(\sqrt{n^2+n} - n)) = \lim_{n \rightarrow \infty} \sin^2(\pi \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}) = \lim_{n \rightarrow \infty} \sin^2(\frac{\pi}{\sqrt{1+1/n}+1}) = 1$.

8. Solve problem 22 on page 63. This is a preview of what you might have in a more advanced course in analysis.

Solution consists of a bunch of (trivial) verifications of the four properties of normed vector spaces. The only trouble you might get into is to forget some of them.

Extra problems.

- 3* I should have included this one into the previous set. Anyway, prove that there exist two irrational numbers a and b such that a^b is rational.

Solution. Clearly, $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$. Take $b = \sqrt{2}$. If b^b is rational, $a = b$ solves the problem. Otherwise, $a = b^b$ does.

- 4* Find two closed subsets $A, B \subset \mathbb{R}$ such that $A + B = \{z = x + y, x \in A, y \in B\}$ is not closed.

Solution. Take $A = \mathbb{N}$ and $B = \{\frac{1}{n} - n, n \in \mathbb{N}\}$. Clearly, $A + B$ contains $\frac{1}{n}$ for all $n \in \mathbb{N}$ but does not contain 0. Hence, it is not closed. However, such an example is not possible if we insist that A is compact.

Homework 2

Problems for all.

1. In the proofs of **F3** and **F4** on pp. 17-18 of your textbook there are similar mistakes. Identify and correct them.

Solution. Uniqueness of $(-a)$ and a^{-1} has to be proved before one can conclude that each of the equations has a unique solution. We can do it simultaneously for all groups. Suppose that b and c satisfy $b \oplus a = a \oplus b = a \oplus c = c \oplus a = id$, where \oplus is the group operation and id is the identity element. Then $b = b \oplus id = b \oplus a \oplus c = id \oplus c = c$.

2. Solve problem 1 on page 29.

Solution. We are to define a field $\mathcal{F} = (\{a, b, c\}, +, \cdot)$. By properties IV and V, it must have a “zero” and a “one” which are not equal to each other (by **F5**). WLOG let $a = 0$ and $b = 1$. Then it is easy to see that the operations must be defined by

$$\begin{array}{cccc} + & a & b & c \\ a & a & b & c \\ b & b & c & a \\ c & c & a & b \end{array} \quad \text{and} \quad \begin{array}{cccc} \cdot & a & b & c \\ a & a & a & a \\ b & a & b & c \\ c & a & c & b \end{array}$$

3. This is a modification of problem 4 on page 29.

- (i) Prove that if $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$ ($a, b, c, d \in \mathbb{R}_+$).

Solution. $\frac{a+c}{b+d} - \frac{a}{b} = \frac{b(a+c) - a(b+d)}{\dots} = \frac{bc - ad}{\dots} = 0$.

- (ii) Use a similar idea to solve problem 4. Do not use a calculator or the standard method for comparing fractions.

Solution. (a) $\frac{223}{71} - \frac{22}{7} = \frac{220+3}{70+1} - \frac{22}{7} < 0$ since $3 < \frac{22}{7}$; (b) $\frac{1351}{780} - \frac{265}{153} = \frac{2702}{1560} - \frac{2650}{1530} = \frac{2650+52}{1530+30} - \frac{265}{153}$. Thus, we need to compare $\frac{52}{30} = \frac{26}{15}$ and $\frac{265}{153}$. Again, $\frac{26}{15} - \frac{265}{153} = \frac{26}{15} - \frac{260+5}{150+3} > 0$ since $\frac{26}{15} > \frac{5}{3} = \frac{25}{15}$. Hence, $\frac{1351}{780} > \frac{265}{153}$.

4. Solve problem 8 on page 30. This is a tedious and rather boring one but you have to go through it once. Sorry about that.

I did go through this once. Not again. :) If you have problems, talk to me.

5. Choose any letter from problem 10 on page 30 and solve it.

Answers. (a) $\sup S = 1$ by **O8**, $\inf S = 0$ by **LUB2**;

(b) $\sup S = \sup\{\frac{1}{3^n} \sum_{k=0}^{n-1} 3^k, n \in \mathbb{N}\} = \sup\{\frac{3^n-1}{2 \cdot 3^n}, n \in \mathbb{N}\} = \frac{1}{2}$, $\inf S = \frac{1}{3}$;

(c) $\inf S = \sqrt{2}$ since the sequence is increasing, $\sup S = 2$ since this is the only solution of the equation $\sqrt{2+a} = a$.

6. Solve problem 11 on page 30.

Solution. We could use the hint but modifying the proof of **LUB1** may be faster. Assume for the contrary that $\sup\{a^n\} = \alpha < \infty$. Then $a^{n+1} < \alpha$ for all $n \in \mathbb{N}$ and $\frac{\alpha}{a}$ must also be an upper bound. But, by **O4**, $\frac{\alpha}{a} < \alpha$. This is a contradiction.

7. Solve problem 14 on page 31.

Hint. Use the result in the next problem and alter the digits in the decimal expansion of a accordingly.

8. Solve problem 15 on page 31.

Solution. Let $a_0.a_1a_2a_3\dots$ be the decimal expansion of a number a . The expansion is periodic if there exists $n, m \in \mathbb{N}$ such that $a_k = a_{k+n}$ for all $k > m$.

Assume that the expansion is periodic. Then

$$a = a_0.a_1a_2a_3\dots a_m + 10^{-m} \sum_{k=1}^{\infty} b \cdot 10^{-nk} = \frac{a_0a_1a_2a_3\dots a_m}{10^m} + \frac{b}{10^m(10^n - 1)}, \quad (0.2)$$

where $b = a_{m+1}a_{m+2}\dots a_{m+n}$. Clearly, this implies that a is rational.

For the converse assume that $a = \frac{p}{q}$ is rational. A way to find the decimal expansion is by long division. But the remainder is always less than p and, therefore, after finitely many steps you are bound to hit the same remainder. At this point the process starts repeating itself.

9. Show that for all $n \in \mathbb{N}$ we have

$$\sum_{k=1}^n k^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

Solution. Step 1. For $n = 1$ the statement is trivially true.

Step 2. Assume that the statement is true for some n . Let us prove it for $n + 1$.

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^n k^3 + (n+1)^3 = \left[\frac{n(n+1)}{2} \right]^2 + (n+1)^3 = \frac{(n+1)^2(4(n+1) + n^2)}{4} = \left[\frac{(n+1)(n+2)}{2} \right]^2.$$

Extra problems.

2* Let $p > 5$ be a prime number. Show that there exists an integer whose digits are all 1's that is divisible by p . (Hint: a solution of problem 15 on page 31 might help you.)

Solution. Observe that (0.2) implies that, for some $c, m, n \in \mathbb{N}$,

$$\frac{1}{p} = \frac{c}{10^m(10^n - 1)} \text{ and, hence, } \frac{9 \cdot 10^m \cdot \overbrace{11\dots 1}^{n \text{ times}}}{p} \in \mathbb{N}.$$

Since $p > 5$ the result follows.

Homework 1

Problems for all.

1. Assume that A , B , and C are subsets of S . Show that:

(a) $c(cA) = A$;

Solution. By definition, $(x \in A)$ iff $(x$ is not in $cA)$ iff $(x$ is in $c(cA))$.

(b) $A \times \emptyset = \emptyset$;

Solution. By definition, $A \times \emptyset = \{(x, y) : x \in A, y \in \emptyset\}$ contains no element and, thus, is equal to \emptyset .

(c) $c(A \cup B \cup C) = cA \cap cB \cap cC$.

Solution. $(x \in c(A \cup B \cup C))$ iff $(x$ is not in $A \cup B \cup C)$ iff $((x$ is not in $A)$ and $(x$ is not in $B)$ and $(x$ is not in $C))$ iff $x \in cA \cap cB \cap cC$.

2. The *symmetric difference* of two sets A and B is defined by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Show that this operation is associative, *i.e.* $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.

Solution (sketch). One can prove the equality by carefully unraveling the definition: $(A \Delta B) \Delta C = ((A \Delta B) \setminus C) \cup (C \setminus (A \Delta B)) = \dots = A \Delta (B \Delta C)$. Some computations can be simplified if you observe that $A \Delta B = (A \cup B) \setminus (A \cap B)$. Here is another way of doing it. Clearly, it is enough to prove the equality for all $x \in A \cup B \cup C = (A \cap B \cap C) \cup (cA \cap B \cap C) \cup (A \cap cB \cap C) \cup (A \cap B \cap cC) \cup (cA \cap cB \cap C) \cup (cA \cap B \cap cC) \cup (A \cap cB \cap cC)$. We can consider each of the above subsets separately and show that $(A \Delta B) \Delta C = (A \cap B \cap C) \cup (cA \cap cB \cap C) \cup (cA \cap B \cap cC) \cup (A \cap cB \cap cC) = A \Delta (B \Delta C)$.

3. Let $f : X \rightarrow Y$ be a function, $A, B \subseteq X$, and $C, D \subseteq Y$.

(i) Prove that

(a) $f(A \cup B) = f(A) \cup f(B)$;

Solution. $f(A \cup B) = \{y \in Y : \text{there exists } x \in A \cup B \text{ such that } y = f(x)\} = \{y \in Y : (\text{there exists } x \in A \text{ such that } y = f(x)) \text{ or } (\text{there exists } x \in B \text{ such that } y = f(x))\} = \{y \in Y : \text{there exists } x \in A \text{ such that } y = f(x)\} \cup \{y \in Y : \text{there exists } x \in B \text{ such that } y = f(x)\} = f(A) \cup f(B)$.

(b) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$;

Solution. $f^{-1}(C \cap D) = \{x \in X : f(x) \in C \cap D\} = \{x \in X : (f(x) \in C) \text{ and } (f(x) \in D)\} = \{x \in X : f(x) \in C\} \cap \{x \in X : f(x) \in D\} = f^{-1}(C) \cap f^{-1}(D)$.

(c) $f^{-1}(f(A)) \supseteq A$.

Solution. $x \in A$ implies $f(x) \in f(A)$ which, in turn, implies $x \in f^{-1}(f(A))$.

(ii) Provide an example when $f(f^{-1}(C))$ is a *proper* subset of C .

Solution. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$. Then $f(f^{-1}(\mathbb{R}_-)) = f(\emptyset) = \emptyset \neq \mathbb{R}_-$.

4. Recall that a function $f : X \rightarrow Y$ is *bijective* if it is one-to-one and onto and *invertible* if there exists an inverse function $f^{-1} : Y \rightarrow X$ such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$. Prove that f is bijective if and only if it is invertible.

Solution. Suppose that f is invertible. Then $f \circ f^{-1} = id_Y$ implies that f is onto and $f^{-1} \circ f = id_X$ implies that f is one-to-one. Indeed, for every $y \in Y$ there

exists $x = f^{-1}(y)$ such that $f(x) = y$ and $f(x_1) = f(x_2)$ implies $x_1 = f^{-1}(f(x_1)) = f^{-1}(f(x_2)) = x_2$. Thus, f is bijective.

Suppose now that f is bijective. Then for every $y \in Y$ there exists (because of surjectivity) a unique (because of injectivity) x such that $f(x) = y$. Thus, we can define $f^{-1}(y) = x$. Clearly, f^{-1} is the inverse function.

5. How many subsets are there of the set $\{1, 2, 3, \dots, n\}$?

Solution. Every element either belongs to a subset or not. Thus, we have 2 choices for n elements. There are 2^n possibilities.

How many maps are there of this set into itself?

Solution. For every element we have n choices. There are n^n possibilities.

How many maps are there of this set onto itself?

Solution. For the first element we have n choices, for the second — $(n - 1)$, for the third — $(n - 2)$, etc. There are $n!$ possibilities.

Extra problems.

- 1* Mathematicians usually don't like to consider sets with elements themselves being sets. One of the reasons is that *the sets of all sets* cannot be well-defined. Why? (Hint: who shaves the barber if he shaves precisely those who don't shave themselves?)

Solution. If the set of all sets \mathfrak{A} were well defined, its subset $S = \{A \in \mathfrak{A} : A \notin A\}$ would also be well defined. But it is not, because $S \in S$ implies $S \notin S$ and vice versa.