

Homework Sets

Math 4121, Spring '06, I. Krishtal

WARNING: “Solutions” given here are, in general, not complete. Some of them would be acceptable if you encountered similar problems in another (more advanced) course but for this course YOUR solutions should be more detailed. If you are using these to prepare for the test it would be a good idea to check that you can supply all the tiny details which I did not spell out here. Solutions below is a Guide, not a Bible.

Homework 10

This is the last homework in the course and it contains miscellaneous exercises referring to what you should already know as well as to the material I will be covering in the remaining 6 lectures. This homework is harder than most of the ones you had before, hence, the deadline is extended until the end of Saturday, 4/22. I will leave an envelope on my office door where it should be put.

Problems for all.

1. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^{-1/2}\chi_{(0,1)}(x)$. Let $\{r_n\}_{n=1}^{\infty}$ be a fixed enumeration of the rationals \mathbb{Q} and

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. Show that, nevertheless, any \tilde{F} such that $\tilde{F} = F$ a.e. is unbounded in any interval.

Solution. By Monotone Convergence Theorem

$$\int F dx = \sum_{n=1}^{\infty} \int 2^{-n} f(x - r_n) dx = \sum_{n=1}^{\infty} \int 2^{-n} f(x) dx = \sum_{n=1}^{\infty} 2^{-n} \int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

Let $I \subset \mathbb{R}$ be an arbitrary open interval and $n \in \mathbb{N}$ be such that $r_n \in I$. For any $M > 0$ we have $\mu(\{x \in I : F(x) > M\}) \geq \mu(\{x \in I : f(x - r_n) > 2^n M\}) > 0$.

2. Let f, g be integrable functions on \mathbb{R}^d . Prove that $f(x - y)g(y)$ is measurable and integrable on \mathbb{R}^{2d} .

Solution. By 10.H, $F(x, y) = f(x - y)$ is measurable. The function $G(x, y) = g(y)$ is measurable, because measurability of $E \subseteq \mathbb{R}$ implies measurability of $\mathbb{R} \times E \subseteq \mathbb{R}^{2d}$. Hence, $H(x, y) = f(x - y)g(y)$ is measurable (see p.13 in Bartle). By Tonelli,

$$\int |f(x - y)g(y)| d\mu(x, y) = \int |g(y)| \left(\int |f(x - y)| dx \right) dy = \|f\|_1 \|g\|_1.$$

3. Let $F(x) = x^2 \sin(1/x^2)$, $x \neq 0$, and $F(0) = 0$. Show that $F'(x)$ exists for every $x \in \mathbb{R}$ but F' is not integrable on $[-1, 1]$.

Solution. Direct computation shows that

$$F'(x) = \begin{cases} 2x \sin(\frac{1}{x^2}) - \frac{1}{2x} \cos(\frac{1}{x^2}), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

One way to show that F' is not integrable is to prove that F does not have bounded variation. Alternatively, since the measure of the set $\{x \in (0, \epsilon) : |\cos(\frac{1}{x^2})| > \frac{1}{2}\}$ is bigger than $\frac{\epsilon}{3}$ for every $\epsilon > 0$ we have that $\int_{-1}^1 |F'(x)| dx \geq \int_0^\epsilon \frac{1}{2x} |\cos(\frac{1}{x^2})| dx = +\infty$.

4. Let F be continuous on $[a, b]$. Show that

$$D^+(F)(x) = \limsup_{h \rightarrow 0^+} \frac{F(x+h) - F(x)}{h} = \lim_{h \rightarrow 0^+} \sup_{t \in (0, h]} \frac{F(x+t) - F(x)}{t}$$

is measurable. Why can one restrict to countably many h in the lim sup above?

Solution. Since F is continuous,

$$\lim_{h \rightarrow 0^+} \sup_{t \in (0, h)} \frac{F(x+t) - F(x)}{t} = \lim_{h \rightarrow 0^+, h \in \mathbb{Q}} \sup_{t \in (0, h]} \frac{F(x+t) - F(x)}{t}.$$

Hence, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \{x \in (a, b) : D^+(F)(x) > \alpha\} &= \bigcap_{h \in \mathbb{Q}_+} \{x \in (a, b) : \sup_{t \in (0, h]} \frac{F(x+t) - F(x)}{t} > \alpha\} = \\ &= \bigcap_{h \in \mathbb{Q}_+} \{x \in (a, b) : \exists y \in (x, x+h] \text{ such that } F(y) - F(x) > \alpha(y-x)\} = \\ &= \bigcap_{h \in \mathbb{Q}_+} \{x \in (a, b) : x \text{ is invisible from the right for } F(\cdot) - \alpha \text{ on } [x, x+h]\}. \end{aligned}$$

This set is G_δ , *i.e.*, a countable intersection of open sets, and, hence, it is measurable. The sets in the curly brackets are open because whenever x is in such a set there will be an interval around it contained in that set.

5. Give an example of an absolutely continuous strictly increasing function $F : [a, b] \rightarrow \mathbb{R}$ such that $F'(x) = 0$ on a set of positive measure. You will need a “fat Cantor set” for that.

Solution. Let C be a fat Cantor set (see one of the extra problems before) and K be the complement of C in $[a, b]$. Define $F : [a, b] \rightarrow \mathbb{R}$ by $F(x) = \int_a^x \chi_K(t) dt$. Clearly, F is absolutely continuous (as an integral) and, if $x > y$, then $F(x) - F(y) = \int_y^x \chi_K(t) dt > 0$ because K is open and C contains no intervals. On the other hand, $F' = \chi_K$ vanishes on a set of positive measure.

Homework 9

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 10.

Problems for all.

Fill in all the details in the proof of Lemma 10.2 and solve problems (10.)H, O, K, S from Bartle's book.

Solutions.

10.2 First, solve 10.D. This follows from

$$\begin{aligned} (A_1 \times B_1) \cup (A_2 \times B_2) &= [((A_1 \cap A_2) \cup (A_1 \setminus A_2)) \times ((B_1 \cap B_2) \cup (B_1 \setminus B_2))] \cup \\ & [((A_1 \cap A_2) \cup (A_2 \setminus A_1)) \times ((B_1 \cap B_2) \cup (B_2 \setminus B_1))] = [(A_1 \cap A_2) \times (B_1 \cap B_2)] \cup \\ & [(A_1 \cap A_2) \times (B_1 \setminus B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \cap B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \setminus B_2)] \cup \\ & [(A_1 \cap A_2) \times (B_2 \setminus B_1)] \cup [(A_2 \setminus A_1) \times (B_1 \cap B_2)] \cup [(A_2 \setminus A_1) \times (B_2 \setminus B_1)]. \end{aligned}$$

Next, prove the first equality in 10.E. This follows from

$$\begin{aligned} (A_1 \times B_1) \setminus (A_2 \times B_2) &= [((A_1 \cap A_2) \cup (A_1 \setminus A_2)) \times ((B_1 \cap B_2) \cup (B_1 \setminus B_2))] \setminus \\ & [((A_1 \cap A_2) \cup (A_2 \setminus A_1)) \times ((B_1 \cap B_2) \cup (B_2 \setminus B_1))] = \left\{ [(A_1 \cap A_2) \times (B_1 \cap B_2)] \cup \right. \\ & \left. [(A_1 \cap A_2) \times (B_1 \setminus B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \cap B_2)] \cup [(A_1 \setminus A_2) \times (B_1 \setminus B_2)] \right\} \setminus \\ & \left\{ [(A_1 \cap A_2) \times (B_1 \cap B_2)] \cup [(A_1 \cap A_2) \times (B_2 \setminus B_1)] \cup [(A_2 \setminus A_1) \times (B_1 \cap B_2)] \right. \\ & \left. \cup [(A_2 \setminus A_1) \times (B_2 \setminus B_1)] \right\} = [(A_1 \cap A_2) \times (B_1 \setminus B_2)] \cup [(A_1 \setminus A_2) \times B_1]. \end{aligned}$$

Finally, apply De Morgan's laws in an obvious way and supply the necessary words which would make the exposition self-contained.

10.H Follows immediately from 2.R (for the first part, you take $\psi = \chi_E$ and $f(x, y) = x - y$).

10.K Since $\chi_D(x, y) = \chi_{\{0\}}(x - y)$, $D \in \mathbf{B} \times \mathbf{B}$. Since the Borel σ -algebra is certainly smaller than the σ -algebra in question, we have that D is measurable. Since $\nu(D_x) \equiv 1$ and $\mu(D^y) \equiv 0$, we have

$$1 = \int \nu(D_x) d\mu \neq \int \mu(D^y) d\nu = 0.$$

10.O After correcting the obvious typo, (a_{mn}) form an infinite matrix with +1 on the main diagonal and -1 on the diagonal above. Therefore, every row in the matrix and every column but first sum to 0. The sum of the elements in the first column is 1. This implies the two equalities in 10.O. The integrability condition in this case would be the absolute convergence of the double sum, which obviously fails.

10.S Here you were supposed to figure out that $\pi(E) = 0$ if and only if E is contained in a countable union of lines parallel to coordinate axes, while $\rho(E) = 0$ if and only if E is contained in a countable union of lines parallel either to one of the coordinate axes or the line $x + y = 0$. As soon as you get this picture all the verifications are trivial.

Extra problems.

10* In the above exercises you have seen a few instances when Tonelli/Fubini fails. Come up with two more, this time on a compact set in \mathbb{R}^2 with the usual Lebesgue measure. The first example should feature a function such that the double integral is infinite but both iterated integrals are finite and equal. In the second example both iterated integrals should be finite but not equal.

Solution. For the first case, we define $f : E = [-1, 1]^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \frac{xy}{(x^2 + y^2)^2},$$

in which case

$$\int_{-1}^1 f(x, y) dx = 0 \text{ a.e.}, \text{ and } \int_{-1}^1 f(x, y) dy = 0 \text{ a.e.}$$

Hence, both iterated integrals are equal to 0. However,

$$\int_E |f(x, y)| dA \geq \int_0^1 \left(\int_0^{2\pi} \frac{|\cos \phi \sin \phi|}{r} d\phi \right) dr = 2 \int_0^1 \frac{dr}{r} = +\infty.$$

For the second case, we define $f : E = [0, 1]^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = \sum_{n=0}^{\infty} 2^{2n} \chi_{(2^{-n-1}, 2^{-n}]}(x) \chi_{(2^{-n-1}, 2^{-n}]}(y) - \sum_{n=1}^{\infty} 2^{2n-1} \chi_{(2^{-n-1}, 2^{-n}]}(x) \chi_{(2^{-n}, 2^{-n+1}]}(y).$$

Drawing a picture, one computes that

$$\int_0^1 \left(\int_0^1 f(x, y) dx \right) dy = 0, \quad \text{but} \quad \int_0^1 \left(\int_0^1 f(x, y) dy \right) dx = 1/4.$$

Homework 8

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 9.

Problems for all.

Solve problems (9.)C, I, M, K, S on pp. 108–112 of Bartle's book.

Solutions.

9.C. For every $M > 0$, we have $(a, a + M] \subset \bigcup (a_n, b_n]$. Hence,

$$\sum_{n=1}^{\infty} l((a_n, b_n]) \geq M,$$

and the sum is infinite.

9.I. In this exercise, I really wanted you to also solve the previous two as well. So I provide the solution of all three.

First, let $l^*(A) < +\infty$ and $E_n = [a_n, b_n)$, $n \in \mathbb{N}$ be such that

$$l^*(A) \leq \sum_{n=1}^{\infty} l((a_n, b_n]) \leq l^*(A) + \epsilon/2.$$

Such a sequence exists almost by definition of l^* . Set $G_\epsilon = \bigcup_{n=1}^{\infty} (a_n, b_n + 2^{-n-2}\epsilon)$ to get $l^*(A) \leq l^*(G_\epsilon) \leq l^*(A) + \epsilon$. Taking $\epsilon = 1/m$, $m \in \mathbb{N}$, we get

$$(*) \quad l^*(A) = \lim_{m \rightarrow \infty} l^*(G_{1/m}),$$

which together with $A \subseteq G_{1/m}$, $m \in \mathbb{N}$, implies the first of the equalities in 9.I.

Next, let $B_n = I_n \setminus A_n$ and $K_\epsilon^n = I_n \setminus G_\epsilon$, where $I_n = [n, n + 1]$ and $A_n = A \cap I_n$. Then $K_\epsilon^n \subseteq B_n$ implies $l^*(K_\epsilon^n) \leq l^*(B_n)$ and

$$l^*(B_n) - \epsilon = 1 - l^*(A_n) - \epsilon \leq 1 - l^*(G_\epsilon) \leq l^*(K_\epsilon^n)$$

proves 9.H. Taking $\epsilon = 2^{-|n|}/m$, $m \in \mathbb{N}$, we get

$$(**) \quad l^*(A) = \lim_{m \rightarrow \infty} \sum_{n=-m}^m l^*(K_{1/m}^n).$$

Since a finite union of compact sets is compact we get the second equality in 9.I.

Now, observe that if $l^*(A) = +\infty$ there is almost nothing to prove. The only thing left is to find a compact subset of A of arbitrarily big measure. This is easily done by taking a sequence of sets of finite measure increasing to A and using what we already proved.

9.K. The first assertion follows immediately from (*) and by taking $B = \bigcap_{m=1}^{\infty} G_{1/m}$. The second assertion follows immediately from (**) and by taking $C = \bigcup_{m,n=1}^{\infty} K_{1/m}^n$.

9.M. (i) follows since any nonempty open set contains an interval; (ii) follows since any compact set in \mathbb{R} is contained in a finite union of intervals of finite length; (iii) is obviously valid for intervals and extends to \mathbf{B} via, say, the first equality in 9.I.

9.S. Follow carefully the proof of lemma 9.3. You will have to use right continuity of g in the end.

Extra problems.

9* For any $f \in L^p(X, \mu)$ define $\lambda_f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\lambda_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}$$

Show that

$$\|f\|_p^p = \int_0^{+\infty} p\alpha^{p-1} \lambda_f(\alpha) d\alpha.$$

Solution. Here is a clever way of proving it. We let $E_\alpha = \{x \in X : |f(x)| > \alpha\}$.

$$\int_0^{+\infty} p\alpha^{p-1} \lambda_f(\alpha) d\alpha = \int_0^{+\infty} \int_X p\alpha^{p-1} \chi_{E_\alpha} d\mu d\alpha = \int_X \int_0^{|f(x)|} p\alpha^{p-1} d\alpha d\mu = \|f\|_p^p,$$

where we used Tonelli's theorem. The function $\lambda_f(\alpha)$ is very important in analysis because it paves way to the study of weak L^p spaces (see, e.g., Folland's book).

Homework 7

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 7 and 8.

Problems for all.

Solve problems (7.)N, Q on p. 78 and (8.)M, N, V on pp. 94–95 of Bartle's book.

Solutions.

7.N. As I indicated in my e-mail, you are supposed to do the following. Let (f_n) be a sequence of functions in M^+ that converges in measure to $f \in M^+$ (which means among other things that these functions are essentially real valued rather than extended real valued). Show that $\int f d\mu \leq \liminf \int f_n d\mu$.

The following is a WRONG proof, but it will do you good to find an error. Clearly, it is enough to show that $f = \liminf f_n$ and apply the usual Fatou's lemma. Let f_{n_k} be a subsequence of the original sequence such that $\lim f_{n_k}(x) = \liminf f_n(x)$ a.e. We still have that f_{n_k} converges to f in measure. Therefore, there is a subsequence of f_{n_k} which converges a.e. to f . But the a.e. limit is a.e. unique. Therefore we must have $f = \liminf f_n$ a.e. Let

And now the proof that is supposed to be correct. Let f_{n_k} be a subsequence of the original sequence such that $\lim \int f_{n_k} d\mu = \liminf \int f_n d\mu$. Let $f_{n_{k_j}}$ be a subsequence of the sequence f_{n_k} which converges to f a.e. (Theorem 7.6) Applying the usual Fatou's lemma we get

$$\int f d\mu = \int \lim f_{n_{k_j}} d\mu \leq \liminf \int f_{n_{k_j}} d\mu = \lim \int f_{n_k} d\mu = \liminf \int f_n d\mu.$$

7.Q. As usually, let $E_{n\alpha} = \{x \in X : |f_n(x) - f(x)| \geq \alpha\}$, $\alpha > 0$, $n \in \mathbb{N}$. Observe that for $x \in E_{n\alpha}$

$$\frac{\alpha}{1 + \alpha} \leq \frac{|f_n(x) - f(x)|}{1 + |f_n(x) - f(x)|} \leq 1.$$

This implies that

$$\mu(E_{n\alpha}) = \int_{E_{n\alpha}} d\mu \leq \frac{1 + \alpha}{\alpha} \int_{E_{n\alpha}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \leq \frac{1 + \alpha}{\alpha} r(f_n - f),$$

which, in turn, yields the first of the required implications. Notice that for this direction you do not need that $\mu(X) < +\infty$. The opposite implication follows from the following inequalities:

$$\begin{aligned} r(f_n - f) &= \int_{E_{n\alpha}} \frac{|f_n - f|}{1 + |f_n - f|} d\mu + \int_{E_{n\alpha}^c} \frac{|f_n - f|}{1 + |f_n - f|} d\mu \leq \\ &\int_{E_{n\alpha}} d\mu + \int_{E_{n\alpha}^c} \frac{\alpha}{1 + \alpha} d\mu \leq \mu(E_{n\alpha}) + \alpha\mu(X). \end{aligned}$$

8.M. In both cases λ is absolutely continuous with respect to μ because $\mu(E) = 0$ implies $E = \emptyset$. Again in both cases $\int_E f d\mu = \sum_{x \in E} f(x)$. In particular, if we assume $\lambda(E) = \int f d\mu$, we get $0 = \lambda(\{x\}) = f(x)$ which yields $\lambda = 0$. This, clearly, is a contradiction. The problem is that μ is not a σ -finite measure.

8.N. By Radon-Nikodym Theorem,

$$\lambda(E) = \int \chi_E d\lambda = \int \chi_E f d\mu.$$

By linearity of an integral we have

$$\int \phi d\lambda = \int \phi f d\mu$$

for all simple functions ϕ . It remains to apply the monotone convergence theorem.

8.V. There are a few little details to check here. First of all, $G(\alpha f + \beta g) = \alpha G(f) + \beta G(g)$ by linearity of the integral. Secondly, $G(f) = 0$ iff $f = 0$ a.e. and, hence, $\|G\| > 0$. Next, G is positive by definition and

$$\left| \int f d\lambda \right| \leq \int |f| d\lambda \leq \int |f| d\nu \leq \left(\int |f|^2 d\nu \right)^{1/2} (\nu(X))^{1/2}$$

implies $\|G\| < +\infty$. By the “baby” RRT, there exists $g \in L^2(\nu)$ such that

$$\int f d\lambda = \int f g d\nu, \quad f \in L^2(\nu).$$

Hence, for $f = \chi_E$, E measurable, $\lambda(E) = \int_E g d\nu$ implies $g \geq 0$. Hence, $\lambda(E) = \int_E g d\lambda + \int_E g d\mu$ and, taking $E = \{x : g(x) \geq 1\}$ or $E = \{x : g(x) = 1\}$, we get $g \leq 1$ ν -a.e. and $\mu\{x : g(x) = 1\} = 0$. Hence, for nonnegative $h \in L^2(\nu)$ we have

$$G(h) = \int h d\lambda = \int h g d\nu = \int h g d\lambda + \int h g d\mu$$

and, consequently,

$$\int h(1 - g) d\lambda = \int h g d\mu.$$

Since all simple functions are in L^2 we get that the above equality for all $h \in M^+$ due to Monotone Convergence Theorem. The rest is in the text.

Extra problems.

8* A function $f : [0, 1] \rightarrow \mathbb{R}$ is called *absolutely continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that any finite collection of mutually disjoint subintervals $(a_k, b_k) \subset [0, 1]$, $k = 1, 2, \dots, n$, for which

$$\sum_{k=1}^n (b_k - a_k) < \delta,$$

satisfies

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \epsilon.$$

Show that every absolutely continuous function is continuous. Provide an example of a uniformly continuous function which is not absolutely continuous.

Comment I will probably find time to do this in class.

Homework 6

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 5 and 6.

The due time for this homework is 11.30 am, March 2 sharp. No exclusions will be made.

Problems for all.

Solve problems (5.)I, O, R on pp. 49–50 and (6.)J, O on pp. 62–63 of Bartle's book.

Solutions.

5.I. If f is integrable, then the real and imaginary parts, $\Re f$ and $\Im f$, are also integrable.

Since $|f| = \sqrt{(\Re f)^2 + (\Im f)^2} \leq 2 \max\{|\Re f|, |\Im f|\}$, $|f|$ is measurable (as a composition of a continuous and measurable functions) and integrable by Corollary (5.4).

Assume now that $|f|$ is integrable. Since f is measurable by assumption, so are $\Re f$ and $\Im f$. Also $|\Re f| \leq |f|$ and $|\Im f| \leq |f|$ imply that $\Re f$ and $\Im f$ are integrable and, hence, f is.

Let $\int f d\mu = r e^{i\theta}$ with $r, \theta \in \mathbb{R}$, $r \geq 0$. Then

$$r = \left| \int f d\mu \right| = \int e^{-i\theta} f d\mu = \int (\Re f \cos \theta + \Im f \sin \theta) d\mu \leq \int |f| d\mu,$$

where we used the following simple relations:

$$|f|^2 - (\Re f \cos \theta + \Im f \sin \theta)^2 = (\Re f \sin \theta - \Im f \cos \theta)^2 \geq 0.$$

5.O. Let $s_m = \sum_{n=1}^m |f_n|$ and $s = \lim s_m = \sum_{n=1}^{\infty} |f_n|$. By MCT,

$$\int s d\mu = \lim_m \int s_m d\mu = \lim_m \sum_{n=1}^m \int |f_n| d\mu = \sum_{n=1}^{\infty} \int |f_n| d\mu < \infty.$$

Hence, s is integrable and can attain infinite value only on a set of measure 0. This implies that $\sum_{n=1}^m f_n \leq s$ converges a.e. and applying LDCT we obtain the desired equality. As I mentioned in class, this is an instance of the celebrated Fubini's theorem which we will learn in full generality towards the end of the course.

5.R. (sketch) First,

$$|f(x, t_1)| \leq |f(x, t_0)| + g(x)|t_1 - t_0|$$

implies that $f(x, t_1)$ is integrable on X . Second,

$$|f(x, t)| \leq |f(x, t_1)| + g(x)|t_1 - t|$$

implies that $f(x, t)$ is integrable on X for every $t \in [a, b]$. Let $F(t) = \int f(x, t) d\mu(x)$. The final step is exactly the same as in the proof of Corollary 5.9.

6.J. Clearly, for $E_n = \{x \in X; n - 1 \leq |f(x)| < n\}$,

$$\chi_{E_1} + \frac{1}{2} \sum_{n=2}^{\infty} n \chi_{E_n} \leq \sum_{n=1}^{\infty} (n-1) \chi_{E_n} \leq |f| \leq \sum_{n=1}^{\infty} n \chi_{E_n}.$$

Integrating with power $p \in [1, \infty)$ we get

$$2^{-p} \sum_{n=2}^{\infty} n^p \mu(E_n) + \mu(E_1) \leq \int |f|^p d\mu \leq \sum_{n=1}^{\infty} n^p \mu(E_n).$$

6.O. It is easily seen that g_0 is measurable and straightforward computation shows that

$$\begin{aligned} \int |g_0|^q d\mu &= \int c^q |f|^{(p-1)q} d\mu = \|f\|_p^{-p} \|f\|_p^p = 1 \quad \text{and} \\ \left| \int f g_0 d\mu \right| &= \int c |f|^p d\mu = \|f\|_p^{p-\frac{p}{q}} = \|f\|_p. \end{aligned}$$

Extra problems.

7* (a) Prove the following Jensen's inequality:

Let (X, \mathbf{X}, μ) be a measure space with $\mu(X) = 1$ and $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then for every integrable function $f \in M^+(X, \mathbf{X}, \mu)$

$$\Phi\left(\int f d\mu\right) \leq \int (\Phi \circ f) d\mu.$$

Solution. By monotone Convergence theorem, it's enough to restrict our attention to step functions. Let $\phi = \sum c_i \chi_{E_i}$ be a step function in standard representation. Using Jensen's inequality for sums (see, e.g. <http://mathcircle.berkeley.edu/trig/node2.html>) we get

$$\Phi\left(\int \phi d\mu\right) = \Phi\left(\sum c_i \mu(E_i)\right) \leq \sum \Phi(c_i) \mu(E_i) = \int (\Phi \circ \phi) d\mu.$$

(b) Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, $g \in L^1(\mathbb{R}^n)$, and $h(x) = f * g(x) = \int f(x-y)g(y)dy$ be the convolution of f and g . Use Jensen's inequality to show that $\|h\|_p \leq \|f\|_p \|g\|_1$.

Solution. For $p = \infty$ the proof is obvious. WLOG $\|g\|_1 = 1$. Then, for $1 \leq p < \infty$ the function $\phi = x^p$ is convex on $[0, \infty)$ and we can use Jensen's inequality with $d\mu = |g(y)|dy$:

$$\begin{aligned} \int \left| \int f(x-y)g(y)dy \right|^p dx &\leq \int \left(\int |f(x-y)g(y)|dy \right)^p dx \\ &\leq \iint |f(x-y)|^p |g(y)| dy dx = \int \int |f(x-y)|^p dx \cdot |g(y)| dy = \|f\|_p^p \cdot \|g\|_1. \end{aligned}$$

Homework 5

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 4.

Problems for all.

Solve problems (4.)J, K, L, O, R on pp. 37–39 of Bartle's book.

Solutions.

- 4.J. (a) Let $f_n = \frac{1}{n}\chi_{[0,n]}$. For every $\epsilon > 0$ and $x \in \mathbb{R}$ there exists $N > \frac{1}{\epsilon}$ such that $|f_n(x)| < \epsilon$ for every $n > N$. Hence, f_n converges to $f = 0$ uniformly. However,

$$0 = \int f d\lambda \neq \lim \int f_n d\lambda = 1.$$

The MCT does not apply because the sequence is not monotone increasing. Fatou's lemma obviously applies.

- (b) Let $g_n = n\chi_{[\frac{1}{n}, \frac{2}{n}]}$, $g = 0$. Again,

$$0 = \int g d\lambda \neq \lim \int g_n d\lambda = 1.$$

However, this time convergence is not uniform (apply the definition). The MCT still does not apply because the sequence is not monotone increasing and Fatou's lemma does apply.

- 4.K. $f \in M^+$ by Corollary 2.10 and the usual properties of the limit. Moreover, for a given $\epsilon > 0$ let $N \in \mathbb{N}$ be such that $\sup |f(x) - f_n(x)| < \epsilon$ for all $n > N$. Then

$$\left| \int f d\mu - \int f_n d\mu \right| \leq \int \epsilon d\mu = \epsilon \mu(X)$$

implies the desired equality.

- 4.L. See Prof. Wilson's handout.

- 4.O. Apply Fatou's lemma to $f_n + h$.

- 4.R. Let ϕ_n be an increasing sequence of real-valued step functions that converges to f pointwise. Let $\phi_n = \sum_{j=1}^{k_n} \lambda_{j,n} \chi_{E_{j,n}}$ be the canonical representation of ϕ_n . Clearly, $\mu(E_{j,n}) < \infty$ for all j, n because f is integrable. Then

$$N = \bigcup_{n \in \mathbb{N}} \{x \in X : \phi_n(x) > 0\} = \bigcup_{n \in \mathbb{N}} \bigcup_{j=1}^{k_n} E_{j,n}$$

implies that N is σ -finite.

Extra problems.

6* Prove the following easy inequality due to Chebyshev:

For $f \in M^+$ and $E_\alpha = \{x \in X : f(x) \geq \alpha\}$,

$$\mu(E_\alpha) \leq \frac{1}{\alpha} \int f d\mu.$$

Solution. Follows from

$$\frac{1}{\alpha} \int f d\mu \geq \frac{1}{\alpha} \int_{E_\alpha} f d\mu \geq \frac{1}{\alpha} \int_{E_\alpha} \alpha d\mu = \mu(E_\alpha).$$

Homework 4

Again, although, I want you to submit only the problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 3.

Problems for all.

Solve problems (3.)C, J, M, Q on pp. 23–26 of Bartle's book and the following problem 3.W.

3.W. Show that a μ -a.e. limit of a sequence of functions is not necessarily unique or measurable.

Solutions.

3.C. This is a standard way of constructing a new measure (metric, norm...) from a sequence of existing ones. Clearly, $\lambda(E) \geq 0$ and $\lambda(X) = \sum_{n=1}^{\infty} 2^{-n} \cdot 1 = 1$. The countable additivity follows from

$$\begin{aligned} \lambda\left(\bigcup_{m=1}^{\infty} E_m\right) &= \sum_{n=1}^{\infty} 2^{-n} \mu_n\left(\bigcup_{m=1}^{\infty} E_m\right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 2^{-n} \mu_n(E_m) = \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} 2^{-n} \mu_n(E_m) = \sum_{m=1}^{\infty} \lambda(E_m). \end{aligned}$$

The interchange of the order of summation is permitted because the double series converges absolutely.

3.J. Recall that $\limsup E_n = \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} E_n \right] := \bigcap_{m=1}^{\infty} F_m$. Since F_m is a decreasing family of subsets with $\mu(F_1) < \infty$, we can apply Lemma 3.4(b) to obtain

$$\mu(\limsup E_n) = \mu\left(\bigcap_{m=1}^{\infty} F_m\right) = \lim_m \mu(F_m) = \lim_m \mu\left(\bigcup_{n=m}^{\infty} E_n\right) \geq \inf_m \sup_{n \geq m} \mu(E_n) = \limsup \mu(E_n).$$

If $\mu(F_m) = \infty$ for all m , the counterexample provided in class still works: for $(\mathbb{R}, 2^{\mathbb{R}}, \mu)$ where μ is the counting measure, and $E_n = [-\frac{1}{n}, \frac{1}{n}]$ we have

$$\limsup \mu(E_n) = \infty \quad \text{but} \quad \mu(\limsup E_n) = 1.$$

3.M. This is a preview of the difference between Borel measurability and Lebesgue measurability, since Lebesgue σ -algebra (measure) is, indeed, the completion of Borel σ -algebra (measure).

Let (X, \mathbf{X}, μ) be a measure space, $\mathbf{Z} = \{F \in \mathbf{X} : \mu(F) = 0\}$, and $\mathbf{X}' = \{E \cup Z, E \in \mathbf{X}, Z \subset F \in \mathbf{Z}\}$ be the completion of \mathbf{X} . We need to show that $\mu' : \mathbf{X} \rightarrow \bar{\mathbb{R}}_+$ defined by $\mu'(E \cup Z) = \mu(E)$ is a measure on (X, \mathbf{X}') . Below are the steps we need to verify:

- μ' is well-defined, *i.e.*, if $E \cup Z = E' \cup Z'$ then $\mu(E) = \mu(E')$.
- μ' is countably additive, *i.e.*,

$$\mu' \left(\bigcup_{n=1}^{\infty} (E_n \cup Z_n) \right) = \sum_{n=1}^{\infty} \mu'(E_n \cup Z_n).$$

To verify the first one, let $F, F' \in \mathbf{Z}$ be such that $Z \subseteq F$ and $Z' \subseteq F'$. Then $E' \subseteq E \cup F$ and $E \subseteq E' \cup F'$, which implies $\mu(E') \leq \mu(E) + \mu(F)$ and $\mu(E) \leq \mu(E') + \mu(F')$. Since $F, F' \in \mathbf{Z}$, we clearly have $\mu(E) = \mu(E')$.

The second one is equally easy:

$$\begin{aligned} \mu' \left(\bigcup_{n=1}^{\infty} (E_n \cup Z_n) \right) &= \mu' \left(\left(\bigcup_{n=1}^{\infty} E_n \right) \cup \left(\bigcup_{n=1}^{\infty} Z_n \right) \right) = \\ \mu \left(\bigcup_{n=1}^{\infty} E_n \right) &= \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \mu'(E_n \cup Z_n), \end{aligned}$$

where we used the fact that a countable union of measure-zero sets also has measure zero.

3.Q. This is one of the first steps on the road leading to the decomposition of charges or functions with bounded variation. I am not sure how much of this road we will be able to cover but it is worth knowing that it exists.

Clearly, the only property to be verified is countable additivity of ν . Let $E = \bigcup_{n=1}^{\infty} E_n = \bigcup_{j=1}^N A_j$. Then sub-additivity follows easily from

$$\begin{aligned} \nu \left(\bigcup_{n=1}^{\infty} E_n \right) &= \sup \sum_{j=1}^N |\mu(A_j)| = \sup \sum_{j=1}^N \left| \mu \left(\bigcup_{n=1}^{\infty} (A_j \cap E_n) \right) \right| = \\ \sup \sum_{j=1}^N \left| \sum_{n=1}^{\infty} \mu(A_j \cap E_n) \right| &\leq \sum_{n=1}^{\infty} \sup \sum_{j=1}^N |\mu(A_j \cap E_n)| = \sum_{n=1}^{\infty} \nu(A_j \cap E_n). \end{aligned}$$

Again, change of the order of summation is permissible because the definition of the charge implies that the series converges unconditionally and, hence, absolutely.

The reverse inequality is trickier. Consider a sequence (a_j) of real numbers such that $a_j < \nu(E_j)$. By the definition of ν , for each $j \in \mathbb{N}$, there exists a finite partition $E = \bigcup A_{i,j}$ such that $a_j \leq \sum_i |\mu(A_{i,j})|$. Observe that $E = \bigcup_{i,j} A_{i,j}$ implies that

$$\sum_j a_j \leq \sum_{i,j} |\mu(A_{i,j})| = \sup_{i,j - \text{finite}} \sum |\mu(A_{i,j})| \leq \nu(E).$$

Finally, taking the sup over all such sequences (a_j) we obtain the desired inequality.

3.W. Follows immediately from an example in 3.V.

Extra problems.

4* Solve problem 3.U. on p. 26 of Bartle's book.

Solution. See, for example, <http://classes.yale.edu/fractals/Labs/PaperFoldingLab/FatCantorSet.html>

- 5* Give an example of an additive but not σ -additive measure, *i.e.*, a function which satisfies Definition 3.1 on p.19 except that formula (3.1) remains true for finite unions and sums but fails for a countable union.

Sketch. Let $X = [0, 1] \cap \mathbb{Q}$, and \mathbf{X} be the restriction of the Borel σ -algebra to X . For $a, b \in X$, define $\mu((a, b) \cap X) = b - a$. It's not too hard to see that μ extends additively but

$$1 = \mu(X) \neq \sum_{r \in \mathbb{Q}} \mu(\{r\}) = 0.$$

Homework 3

This set is the first from the Bartle's book. Although, I want you to submit only the 5 problems below (+ the extra, if you want), I strongly recommend to you to look at ALL the problems after chapter 2.

Problems for all.

Solve problems (2.)B, H, K, O, P on pp. 14–17 of Bartle's book.

Solutions.

2.B. Follows from the fact that

$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] \quad \text{and} \quad (a, b] = (a, +\infty) \cap (b, +\infty)^c.$$

2.H. By definition,

$$A = \liminf A_n = \bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} A_n \right] \quad \text{and} \quad B = \limsup A_n = \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} A_n \right].$$

We have $(x \in A) \Rightarrow ((\exists m \in \mathbb{N})(\forall n \geq m)(x \in A_n)) \Rightarrow ((\forall m \in \mathbb{N})(x \in \bigcup_{n=m}^{\infty} A_n))$. Hence, $A \subseteq B$. The sequence $A_n = \mathbb{N} \setminus \{n\}$ is a non-monotonic example of $A = B = \mathbb{N}$. The sequence $A_n = 2\mathbb{N}$ if n is even and $A_n = 2\mathbb{N} - 1$ if n is odd gives $A = \emptyset$ and $B = \mathbb{N} = X$.

2.K. We need to calculate the inverse images $\mathcal{A}_\alpha = \{x \in X : f_A(x) > \alpha\}$. We have

$$\mathcal{A}_\alpha = \begin{cases} X, & \alpha < -A; \\ f^{-1}((\alpha, +\infty)), & -A \leq \alpha < A; \\ \emptyset, & A \leq \alpha. \end{cases}$$

Clearly, these sets are measurable.

2.O. We first solve the preceding exercise. Let $\mathbf{Y} = \{E \subseteq Y : f^{-1}(E) \in \mathbf{X}\}$. Since $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$, we have that both \emptyset and Y are in \mathbf{Y} . Next, if $E \in \mathbf{Y}$, $f^{-1}(E^c) = (f^{-1}(E))^c \in \mathbf{X}$. Finally, if $E_n \in \mathbf{Y}$, $n \in \mathbb{N}$, we have $f^{-1}(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} f^{-1}(E_n) \in \mathbf{X}$. This implies that \mathbf{Y} is a σ -algebra. Now F is in the smallest σ -algebra that contains \mathbf{A} . Therefore, $F \in \mathbf{Y}$ and the statement follows.

2.P. Follows immediately from 2.B and 2.O.

Extra problems.

3* Solve problem 2.W. on p. 18 of Bartle's book.

Solution. Let \mathbf{A} consist of $A_n = [n, +\infty)$. Then $\mathbf{A} \neq \mathbf{M}$ because $\emptyset = \bigcap_{n=1}^{\infty} A_n \notin \mathbf{A}$ and $\mathbf{M} \neq \mathbf{S}$ because $A_n^c \notin \mathbf{M}$ for any $n \in \mathbb{N}$.

Homework 2

Problems for all.

This set seems to be easier than the first one but contains many facts and examples that I feel necessary for you to know. In the last 3 problems do not forget to prove that the examples you provide work.

1. Solve problem 17 on p.133 of Rosenlicht's book. You only need to add a few words to the "proof" you should have had in a calculus course.

Solution. Since u' and v' are continuous, the integrals on both sides make sense and the FTC applies to the function $f(x) = u(x)v'(x) + v(x)u'(x) = \frac{d}{dx}(u(x)v(x))$:

$$\int_a^b f(x)dx = \int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx = u(b)v(b) - u(a)v(a).$$

2. Solve problem 20 on p.134 of Rosenlicht's book. I'm afraid, we won't have many opportunities in this course to explore geometric applications of integral calculus - so let us seize one when it presents itself.

Solution. By definition, the length $\ell(f)$ of the curve defined by the vector function f is

$$\begin{aligned} \ell(f) &= \sup_P \sum_{i=1}^N d(f(x_{i-1}), f(x_i)) = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^N \sqrt{\sum_{j=1}^n (f_j(x_i) - f_j(x_{i-1}))^2} = \\ &= \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^N \sqrt{\sum_{j=1}^n ((f'_j(x_{ij}^*))\Delta x_i)^2} = \lim_{\Delta x_i \rightarrow 0} \sum_{i=1}^N \sqrt{\sum_{j=1}^n (f'_j(x_i) + o(1))^2 \Delta x_i} = \\ &= \lim_{\Delta x_i \rightarrow 0} \left(\sum_{i=1}^N \left(\sqrt{\sum_{j=1}^n (f'_j(x_i))^2 + o(1)} \right) \Delta x_i \right) = \int_a^b \sqrt{\sum_{j=1}^n (f'_j(x))^2} dx, \end{aligned}$$

where we used the triangle inequality to replace the "sup" with the "lim", the celebrated Mean Value Theorem, and, finally, boundedness and the uniform(!) continuity of the integrand to justify our manipulations with $o(1)$.

3. Solve problem 21(a) on p.134 of Rosenlicht's book. Hint: express the sum as a Riemann sum of a certain integral.

Solution. The sum in the limit is precisely the upper D'Arboux sum for the function $f : [0, 1] \rightarrow \mathbb{R}$, $f(x) = x^k$ associated with the uniform n -element partition of $[0, 1]$. Hence,

$$\lim_{n \rightarrow \infty} \frac{1^k + 2^k + \dots + n^k}{n^{k+1}} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \left(\frac{j}{n}\right)^k = \int_0^1 x^k dx = \frac{1}{k+1}.$$

4. Solve problem 1 on p.160 of Rosenlicht's book. This is an easy example.

Solution. Consider $f_n(x) = |x|^{\frac{1}{n}}$. Clearly,

$$\lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} f_n(x) = 1 \neq 0 = \lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} f_n(x).$$

5. Solve problem 3 on p.161 of Rosenlicht's book. This one is slightly harder.

Solution. Consider the sequence of "hat" functions defined on p. 138 of Rosenlicht's book

$$f_n = \begin{cases} 4n^2x, & 0 \leq x \leq \frac{1}{2n}; \\ 4n - 4n^2x, & \frac{1}{2n} < x \leq \frac{1}{n}; \\ 0, & \frac{1}{n} < x \leq 1. \end{cases}$$

and let $g_n = nf_n$. Clearly, $g_n \rightarrow 0$ pointwise but $\int g_n = n$.

6. Solve problem 5 on p.161 of Rosenlicht's book. This might be tricky but not too much so.

Solution. Let $f : [0, 1] \rightarrow \mathbb{R}$ be the Riemann function defined on p.90, Problem 1(d), of Rosenlicht's book (see also the previous homework set). Define the sequence $f_n = f^{\frac{1}{n}}$. These functions are all integrable (see the proposition about composite functions that we proved in class). However the pointwise limit is the Dirichlet function which is discontinuous at every point and, therefore, not (Riemann) integrable.

Extra problems.

Let's see if you are up to Bartle's hope (see p.2).

- 2* Let $f_n = \frac{e^{-nx}}{\sqrt{x}}$. Show that $f_n \rightarrow 0$ pointwise but not uniformly on $(0, +\infty)$. Show that, nevertheless, $I_n \rightarrow 0$, where $I_n = \int_0^\infty f_n(x)dx$.

Sketch. The convergence $f_n \rightarrow 0$ is, indeed, not uniform because

$$\lim_{x \rightarrow 0^+} f_n(x) = \infty$$

for all $n \in \mathbb{N}$. Therefore, for every $N \in \mathbb{N}$ there exists $x \in (0, \infty)$ such that $f_N(x) > 1$. This, obviously, contradicts the definition of uniform convergence. However, using the change of variable formula, we get

$$I_n = \int_0^\infty f_n(x)dx = \frac{1}{\sqrt{n}} \int_0^\infty \frac{e^{-y}}{\sqrt{y}} dy \leq \frac{1}{\sqrt{n}} \left(\int_0^1 \frac{1}{\sqrt{y}} dy + \int_1^\infty e^{-y} dy \right) = \frac{1}{\sqrt{n}}(2+e^{-1}) \rightarrow 0.$$

Homework 1

Problems for all. Last time the first homework was a warm-up. This one is a bucket of cold water. :) Unlike the exam situation, however, I promise to help.

1. Let f be (Riemann) integrable and concave on $[a, b]$. Show that

$$(b-a)\frac{f(a)+f(b)}{2} \leq \int_a^b f(x)dx \leq (b-a)f\left(\frac{a+b}{2}\right).$$

Solution. Let us first prove the inequality

$$(b-a)\frac{f(a)+f(b)}{2} \leq \int_a^b f(x)dx.$$

Define $\phi : [a, b] \rightarrow \mathbb{R}$ by $\phi(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. By concavity, $\phi \leq f$. Hence,

$$\int_a^b \phi(x)dx = (b-a)\frac{f(a)+f(b)}{2} \leq \int_a^b f(x)dx.$$

To prove the second inequality we use a different method (there might be easier ones but I thought it would be useful for you to see the one below).

Since f is concave and integrable, we can use a specific sequence of Riemann sums to obtain

$$\begin{aligned} \int_a^b f(x)dx &= \lim_{n \rightarrow \infty} \sum_{k=1}^{2n} f\left(a + \left(k - \frac{1}{2}\right)\frac{b-a}{2n}\right) \frac{b-a}{2n} = \\ \lim_{n \rightarrow \infty} \frac{b-a}{2n} \sum_{k=1}^n \left[f\left(a + \left(k - \frac{1}{2}\right)\frac{b-a}{2n}\right) + f\left(a + \left(2n - k + \frac{1}{2}\right)\frac{b-a}{2n}\right) \right] &\leq \\ \lim_{n \rightarrow \infty} \frac{b-a}{2n} \sum_{k=1}^n 2f\left(\frac{a+b}{2}\right) &= (b-a)f\left(\frac{a+b}{2}\right). \end{aligned}$$

The points in the Riemann sum are computed as midpoints of subintervals of length $\frac{b-a}{2n}$ that partition $[a, b]$.

2. This one is in a sense a preview of one of the main theorems on Riemann integrability that will be covered in class. If the theorem is covered before this homework is due, do not just quote it. Prove the statement below with “bare hands”.

Let f be the Riemann function defined on p.90, Problem 1(d), of Rosenlicht’s book. Show that f is integrable on any finite interval. Can you compute the integral? (You can assume that the set \mathbb{Q} is countable.)

Solution. From the solution of the cited problem (see last semester’s homework), we know that the set of discontinuities of f is precisely \mathbb{Q} . Since the set of rationals is countable, it has measure 0. This implies integrability of f . To see this directly, we will show that for any fixed $\epsilon > 0$ there exists a partition of $[0, 1]$ such that $\mathcal{U}(P, f) < \epsilon$ (WLOG, we can assume $[a, b] = [0, 1]$). Since $\mathcal{L}(P, f) = 0$ for all P , this would imply integrability of f .

From the definition of f we see that there are only finitely many points $y_i, i = 1, \dots, N$, for which $f(y_i) > \frac{\epsilon}{2}$. Let these points be centers of the intervals $I_i = (y_i - 2^{-i-2}\epsilon, y_i + 2^{-i-2}\epsilon)$ and P be a partition containing the end points of these intervals. Then

$$\mathcal{U}(P, f) \leq \frac{\epsilon}{2} + \sum_{i=1}^N 2^{-i-1}\epsilon < \epsilon.$$

3. Solve problem 8 on p.133 of Rosenlicht's book. This may help you to solve the extra problem.

Solution. First of all observe that since f is defined and monotonic on the *closed* interval $[a, b]$, it is bounded. Again WLOG $[a, b] = [0, 1]$ and f is increasing. For each $n \in \mathbb{N}$ let us choose a uniform partition $P_n = \{x_j = \frac{j}{n}, j = 0, \dots, n\}$. Then

$$\mathcal{U}(P_n, f) = \frac{1}{n} \sum_{j=1}^n f(x_j) \quad \text{and} \quad \mathcal{L}(P_n, f) = \frac{1}{n} \sum_{j=1}^n f(x_{j-1}).$$

Therefore, if $|f(x)| \leq M$ for all $x \in [0, 1]$ we have

$$\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = \frac{f(x_n) - f(x_0)}{n} \leq \frac{2M}{n}.$$

It remains to choose n sufficiently large.

4. Solve problem 13 on p.133 of Rosenlicht's book. You may want to look at problem 12 to get some ideas. Some of you may notice some connection with the Gelfand's spectral radius formula (don't worry, if you've never heard about it before).

Solution. Above I mentioned the spectral radius formula, but a more significant reason to look at this formula is the fact (probably, known to some of you) that the integrals on the left hand side of the formula gives rise to norms in L^p spaces for $p = n \in \mathbb{N}$, while the formula on the right hand side is essentially the L^∞ norm. We will learn quite a bit about these spaces later in this course.

First observe that

$$\left(\int_a^b (f(x))^n dx \right)^{\frac{1}{n}} \leq \left(\int_a^b \max_{y \in [a, b]} (f(y))^n dx \right)^{\frac{1}{n}} = (b-a)^{\frac{1}{n}} \max_{x \in [a, b]} (f(x))$$

implies LHS \leq RHS. On the other hand, for a given $\epsilon > 0$ let $[c, d] \subset [a, b]$ be such that

$$\max_{y \in [a, b]} (f(y)) - f(x) \leq \epsilon$$

for all $x \in [c, d]$. Then

$$\left(\int_a^b (f(x))^n dx \right)^{\frac{1}{n}} \geq \left(\int_c^d (f(x))^n dx \right)^{\frac{1}{n}} \geq (d-c)^{\frac{1}{n}} (\max_{y \in [a, b]} f(y) - \epsilon)$$

implies RHS \leq LHS.

5. Solve problem 16 on p.133 of Rosenlicht's book. This is another one of my numerous attempts to make you comfortable with abstract material. To me, this is the easiest problem in this set.

Solution. We need to show that for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\|f - g\|_{C([a,b])} < \delta$ we have $\left| \int_a^b f(x)dx - \int_a^b g(x)dx \right| < \epsilon$. Since $\|f - g\|_{C([a,b])} = \max_{x \in [a,b]} |f - g|(x)$ and $\left| \int_a^b (f(x) - g(x)) \right| \leq \int_a^b |f - g|(x)dx$, the result follows.

Extra problems.

This time, a relatively easy one.

- 1* Solve problem 1 without assuming integrability of f .

Comment. It's enough to observe that $[a, b]$ can be decomposed on at most two intervals on each of which f is monotonic.