

Final Exam

No notes, books or calculators allowed. Read carefully the statement of the problem, before you start to work. Please, show your work and justify your answers in order to get proper credit.

(1) [16 Pts.] Compute derivatives of the following functions:

$$(a) f(x) = 25x^{\ln 5}; \quad (b) f(x) = \arccos(5 \cos x);$$

$$(c) f(x) = e^{\sin(x^2 - \ln x)}; \quad (d) f(x) = \cosh^{-1}\left(\frac{e^x + e^{-x}}{2}\right).¹$$

$$(a) f'(x) = 25(\ln 5)x^{\ln 5 - 1}.$$

$$(b) f'(x) = \frac{5 \sin x}{\sqrt{1 - 25 \cos^2 x}}.$$

$$(c) f'(x) = \left(2x - \frac{1}{x}\right) \cos(x^2 - \ln x) e^{\sin(x^2 - \ln x)}.$$

$$(d) f'(x) = \frac{d}{dx} \left(\cosh^{-1}(\cosh x) \right) = 1.$$

(2) [15 Pts.] Evaluate the following indefinite integrals:

$$(a) \int \cos x \sin x \, dx; \quad (b) \int x e^x \, dx; \quad (c) \int \frac{x^2 + 2x - 1}{x^2 - 1} \, dx.$$

$$(a) \int \cos x \sin x \, dx = \frac{1}{2} \int \sin 2x \, dx = -\frac{1}{4} \cos 2x + C = \frac{1}{2} \sin^2 x + C_1 = -\frac{1}{2} \cos^2 x + C_2.$$

$$(b) \int x e^x \, dx = x e^x - \int e^x \, dx = (x - 1)e^x + C.$$

$$(c) \int \frac{x^2 + 2x - 1}{x^2 - 1} \, dx = \int \left(1 + \frac{2x}{x^2 - 1}\right) \, dx = x + \ln |x^2 - 1| + C.$$

¹If you don't remember what is \cosh^{-1} , don't worry - the explicit formula is not really needed here.

(3) [15 Pts.] Consider the following equation $1 + 2x^3 + 0.0001x^{234567} = 0$

(a) State the theorem, which allows you to conclude that this equation has at least one solution.

(b) Use the increasing/decreasing test to show that the above solution is unique. Do not try to compute the solution.

(a) **Intermediate Value Theorem.** If a function f is continuous on $[a, b]$ and $f(a) < f(b)$ then for each y in $[f(a), f(b)]$ there exists c in $[a, b]$ such that $f(c) = y$.

This theorem can be used because

$$\lim_{x \rightarrow -\infty} (1 + 2x^3 + 0.0001x^{234567}) = -\infty \text{ and } \lim_{x \rightarrow \infty} (1 + 2x^3 + 0.0001x^{234567}) = \infty.$$

It turns out that **Rolle's Theorem** also yields needed result if we consider an antiderivative F of f ($f(x) = 1 + 2x^3 + 0.0001x^{234567}$) and observe that

$$\lim_{x \rightarrow -\infty} F(x) = \lim_{x \rightarrow \infty} F(x) = \infty.$$

I must admit though, that no mathematician in his right mind would think of using Rolle's theorem here.

(b) Observe that $f'(x) = 6x^2 + 23.4567x^{234566}$ is a non-negative function for all (real) x . Hence, I/D test implies that f is an increasing function. Therefore, it can only have 1 root.

(4) [18 Pts.] Evaluate the following integrals (warning: some of them might be improper integrals):

$$(a) \int_{e^{-1}}^e |\ln x| dx; \quad (b) \int_0^1 \frac{\arctan x}{x^2 + 1} dx; \quad (c) \int_1^{-1} \frac{1}{x^4} dx.$$

$$\begin{aligned} (a) \int_{e^{-1}}^e |\ln x| dx &= - \int_{e^{-1}}^1 \ln x dx + \int_1^e \ln x dx = -x(\ln x - 1) \Big|_{e^{-1}}^1 + x(\ln x - 1) \Big|_1^e \\ &= -2e^{-1} - (-1) + 0 - (-1) = 2(1 - e^{-1}). \end{aligned}$$

$$(b) \int_0^1 \frac{\arctan x}{x^2 + 1} dx = \frac{1}{2} \arctan^2 x \Big|_0^1 = \frac{1}{2} \left(\frac{\pi}{4}\right)^2 - 0 = \frac{\pi^2}{32}.$$

$$\begin{aligned} (c) \int_1^{-1} \frac{1}{x^4} dx &= - \int_{-1}^0 \frac{1}{x^4} dx - \int_0^1 \frac{1}{x^4} dx = - \lim_{\epsilon \rightarrow 0^-} \int_{-1}^{\epsilon} \frac{1}{x^4} dx - \lim_{\delta \rightarrow 0^+} \int_{\delta}^1 \frac{1}{x^4} dx \\ &= \frac{1}{3} \lim_{\epsilon \rightarrow 0^-} \left[\frac{1}{x^3}\right]_{-1}^{\epsilon} + \frac{1}{3} \lim_{\delta \rightarrow 0^+} \left[\frac{1}{x^3}\right]_{\delta}^1 = \frac{1}{3} \lim_{\epsilon \rightarrow 0^-} (\epsilon^{-3} - 1) + \frac{1}{3} \lim_{\delta \rightarrow 0^+} (1 - \delta^{-3}). \end{aligned}$$

These limits do not exist (are infinite). Therefore, this improper integral diverges. Compare with (6)(b).

(5) [11 Pts.] Find the area of the region bounded by the curves $x^2 = y$ and $y^2 = x$.

These curves intersect each other when $x = 0$ and $x = 1$. Therefore the area of the region is given by

$$\int_0^1 |x^2 - \sqrt{x}| dx = \int_0^1 (x^{0.5} - x^2) dx = \frac{2}{3} - \frac{1}{2} = \frac{1}{3}$$

(6) [20 Pts.] Decide whether the following statements are true (T) or false (F).

(a) If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = K$, then $\lim_{x \rightarrow a} (5f(x) - 3g(x)) = 5L - 3K$.

True. See the properties of the limits.

(b) If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$, then $\lim_{x \rightarrow a} f(x) - g(x) = \infty$.

False. Take for example $g(x) = f(x)$. In fact, the last limit can be anything.

(c) If $f(x) \geq g(x)$ when $0 < x < 1$, then $\lim_{x \rightarrow 0^+} f(x) \geq \lim_{x \rightarrow 0^+} g(x)$.

False. The limits may not exist.

(d) If $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$ then the function $f(x)$ is continuous at $x = a$.

False. The function may not even be defined at $x = a$.

(e) If a function $f(x)$ is twice differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

True. See the properties of differentiable functions.

(f) If a function f is differentiable on (a, b) and $f'(c) = 0$ at some point c in (a, b) , then f has a local maximum or minimum at c .

False. Take $f(x) = x^3$ and $c = 0$.

(g) If a function f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

True. Mean value theorem. In fact, c belongs to the open interval (a, b) .

(h) If f is differentiable and $f'(x) = g'(x)$, then $f(x) = g(x)$.

False. Take $f(x) - g(x) = c$.

(i) If f is continuous and positive on $[a, b]$, then $\int_a^b f(x) dx > 0$.

True.

(j) If f is even and continuous on $[-a, a]$, then $\int_{-a}^a f(x) dx = 0$.

False. This is true for an odd function.

(7) [5 pts.] State the theorem, which you think to be the most important in the course.

You can do that yourself, don't you?