M4121, MIDTERM TEST
March 2–20, 2005.

Instructions: Don't be afraid of the number of problems on the test. You are supposed to do the first 3 problems and then choose another 3 from the rest. If you want to do the other problems, you can, but only 6 problems will count towards your grade. You are supposed to work on the test by yourself but you can always ask me for help. I might give you some hints, although I will be less inclined to do it than usually.

I do not expect you to work during the Spring Break, but if you want, you can. The test should be returned to me or to Cupples I, Room 100 by **2.00 PM, March 20.** *Since you have that much time to do the test, no extensions will be granted. I would also ask you to exercise your respect and present clearly legible solutions.*

Obligatory Problems — Do ALL

1. (15pt) Present a careful construction of the Riemann integral on a bounded domain in $\mathbb{R}^2$. You may use the concept of null sets.

   Comments. This was an exercise for your benefit rather than to check you. I graded it more like an essay. I paid most of my attention to how you treated the case of the general domain and its boundary being a null set (this is not the same as the boundary being a Jordan curve).

2. (20pt) Let $(X, \mathbf{X})$ be a measurable space and $\Phi : \mathbf{X} \to \mathbb{R}$ be a charge. Show that

   (a) $\sup\{|\Phi(A)|;\, A \in \mathbf{X}\} < \infty$.

   (b) There exist two finite measures Φ^+ and Φ^- such that $\Phi = \Phi^+ - \Phi^-$.

   (c) The total variation $|\Phi|$ of Φ defined in the h/w exercise 3.Q satisfies $|\Phi| = \Phi^+ + \Phi^-$.

   Comments. There are two good ways to prove 2(a), one uses the Hahn Decomposition and the other does not. The second one is lengthy because of the following reason.

   Suppose $\sup\{|\Phi(A)|;\, A \in \mathbf{X}\} = \infty$. Since $|\Phi(X)| < \infty$, we have $\sup\{\Phi(A);\, A \in \mathbf{X}\} = \infty$. Let $A_n \in \mathbf{X}$ be a sequence such that $\Phi(A_n) > n$. Observe that it is NOT true that you can assume this sequence to be increasing. The sequence $B_n = \bigcup_{k=1}^n A_k \in \mathbf{X}$ is increasing but you can't easily show that $\sup\{\Phi(B);\, B \in \mathbf{X}\} = \infty$ just as you do for positive sets. There is a way around but it's ugly and none of you actually was able to pull it through.

An easy way to prove 2(a) is the following. Let $X = P \cup N$ be a Hahn decomposition for Φ. If $E \in \mathbf{X}$ then $\Phi(E) = \Phi(E \cap P) + \Phi(E \cap N) \leq \Phi(E \cap P) \leq \Phi(E \cap P) + \Phi(P \backslash (E \cap P)) = \Phi(P)$ implies that $\sup\{\Phi(A);\, A \in \mathbf{X}\} = \Phi(P) < \infty$.

As most of you noticed, 2(b) is just the Jordan Decomposition, one only needed to convince me that what you get are indeed finite measures.

In 2(c) you were supposed to show that two different ways of obtaining the Jordan Decomposition are equivalent, *i.e.*

$$\Phi(E \cap P) - \Phi(E \cap N) = \sup \sum_j |\Phi(A_j)|,$$

where the "sup" is taken over all finite partitions of E. The $\leq$ inequality is immediate and the other one follows from

$$\sum_j |\Phi(A_j)| = \sum_j |\Phi(A_j \cap P) + \Phi(A_j \cap N)| \leq$$

$$\sum_j \Phi(A_j \cap P) - \sum_j \Phi(A_j \cap N) = \Phi(E \cap P) - \Phi(E \cap N).$$

3. (20pt) Let $(X, \mathbf{B}, \mu)$ be a measure space, where $\mathbf{B}$ is the (Borel) σ-algebra generated by open subsets of X and the measure μ satisfies

$$\mu(M) = \inf\{\mu(G),\, M \subseteq G,\, G \text{ open}\}.$$

Show that the set of continuous real-valued functions on X is dense in $L^1(X, \mathbf{B}, \mu)$, *i.e.*, any L^1 function can be approximated in L^1-norm by continuous functions. If the general case is too hard for you, do the case $(X, \mathbf{B}, \mu) = (\mathbb{R}, \mathbf{B}, \lambda)$, where λ is the Lebesgue measure. This case will be graded out of 15pt.

Solution. As some of you noticed, I did not specify the topological restrictions on the space X. Since we did not explicitly study topological spaces in this sequence, the assumption was that the space X is a metric space. I apologize for not stating this explicitly. In fact, the result is still true for a large class of topological spaces, you can find it in most of the standard textbooks (*e.g.* W.Rudin, *Real and Complex Analysis*, 1966; Theorems 2.14, 2.23, 3.14). I'm surprised that not many of you did that.

Again, there are at least two ways of solving the problem. The first uses Lusin's theorem and can be found in the above reference. Below is the sketch of the solution for metric spaces that you might have come up with.

i. Simple functions are dense in L^1 by definition of the integral. Hence, it is enough to approximate a characteristic function of any measurable set by continuous functions in L^1 norm to prove the result.

ii. By assumption on μ, it follows immediately that for every measurable set E and any $\epsilon > 0$ there exist a closed set F_E and an open set G_E such that

$$F_E \subseteq E \subseteq G_E \quad \text{and} \quad \mu(G_E) - \mu(F_E) < \epsilon.$$

iii. Let ρ be a metric on X and $\rho(x, A) = \inf\{\rho(x, y), y \in A\}$. Then

$$\phi(x) = \frac{\rho(x, X\backslash G_E)}{\rho(x, X\backslash G_E) + \rho(x, X\backslash F_E)}$$

is continuous (as per a h/w exercise last semester), $\phi(F_E) = \{1\}$, $\phi(G_E) = \{0\}$, and $\phi(X) = [0, 1]$. Hence, $|\chi_E - \phi| \leq 1$ and $\text{supp}(\chi_E - \phi) \subseteq G_E\backslash F_E$. Clearly, this implies

$$\int |\chi_E - \phi| d\mu < \epsilon.$$

Choice Problems — Select Three

4. (15pt) Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable on compact intervals and satisfy $\lim\limits_{x\to+\infty} f(x) = c$. Show that $\lim\limits_{x\to+\infty} \frac{1}{x}\int_0^x f(t)dt = c$.

   Sketch. This is a usual "good guy – bad guy" problem. On one side

$$\lim_{x\to+\infty} \frac{1}{x}\int_0^N f(t)dt = 0$$

for every $N \in \mathbb{R}$. On the other side, for large enough N, we have $|f(t) - c| < \epsilon$ for all $t \geq N$ and the estimate

$$\left|\frac{1}{x}\int_N^x f(t)dt - c\right| \leq \frac{1}{x}\int_N^x |f(t) - c|dt + \frac{|c|N}{x - N} < \epsilon + \frac{|c|N}{x - N}$$

is enough to finish the proof.

5. (15pt) Let $f \in M^+(X, \mathbf{X})$ satisfy $\int f d\mu < +\infty$. Show that for every $\epsilon > 0$ there exists $E \in \mathbf{X}$ such that $\mu(E) < +\infty$ and $\int f d\mu \leq \int\limits_E f d\mu + \epsilon$.

   Solution. Let $\phi \in M^+$ be a simple function such that $\phi \leq f$ and $\int f d\mu - \int \phi d\mu < \epsilon$. Such a function exist by definition of an integral and, since a simple function attains only finitely many values and its integral is finite, we have $\int \phi d\mu = \int_E \phi d\mu$ for some E with $\mu(E) < +\infty$. The inequalities

   $$\int f d\mu \geq \int_E f d\mu \geq \int_E \phi d\mu = \int \phi d\mu$$

   finish the proof.

6. (15pt) Prove the following version of the Monotone Convergence Theorem (also due to B. Levi). Let (f_n) be a monotone increasing sequence of integrable functions such that $\int f_n d\mu \leq K$. Then there exists an integrable function f such that (f_n) converges to f almost everywhere and

   $$\lim_n \int f_n d\mu = \int f d\mu.$$

   Solution. Since $f_n - f_1 \geq 0$, we may WLOG assume that $f_n \in M^+$. We can define $\bar{f} \in M^+$ as a pointwise limit of f_n, which exists by monotonicity (we can let $\bar{f}(x) = +\infty$ if $f_n(x)$ is unbounded). By the version of MCT in Bartle, which now applies, $\lim_n \int f_n d\mu = \int \bar{f} d\mu$. The assumptions of the theorem imply that this limit is not greater than K. Hence, the set $A = \{x : \bar{f}(x) = +\infty\}$ has measure zero and we can redefine f on A to obtain a real valued function that satisfies the conclusion of the theorem. Observe that proving that $\mu(A) = 0$ is essentially equivalent to proving MCT. A number of you presented an invalid proof of this fact and lost quite a few points for that.

7. (15pt) Let $p \in [1, \infty]$. Find an example of a measure space $(X, \mathbf{B}, \mu)$ and a function f on X such that $f \in L^p$ but $f \notin L^q$ for all $q \neq p$, $q \in [1, \infty]$. Don't forget the proof.

   Solution. First of all, what the problem asks for is a family of functions f_p with the stated property. Supplying $f_\infty = const$ was not enough to get full credit. An inspiration for $p \neq \infty$ can be found in exercise 6.M. Consider $X = (0, \infty)$ with Lebesgue measure and let

   $$f_p(x) = x^{-1/p}(1 + |\log x|)^{-2/p}.$$

The following computations supply the necessary proof:

$$\int |f_p|^p d\mu = \int_0^\infty x^{-1}(1+|\log x|)^{-2} dx =$$

$$\int_0^1 x^{-1}(1-\log x)^{-2} dx + \int_1^\infty x^{-1}(1+\log x)^{-2} dx$$

$$= \int_{-\infty}^0 \frac{du}{(1-u)^2} + \int_0^\infty \frac{du}{(1+u)^2} = \int_1^\infty \frac{2du}{u^2} < +\infty;$$

$$\int |f_p|^q d\mu = \int_0^\infty x^{-q/p}(1+|\log x|)^{-2q/p} dx =$$

$$\int_0^1 x^{-q/p}(1-\log x)^{-2q/p} dx + \int_1^\infty x^{-q/p}(1+\log x)^{-2q/p} dx$$

$$= \int_{-\infty}^0 \frac{e^{u(1-q/p)}}{(1-u)^2} du + \int_0^\infty \frac{e^{u(1-q/p)}}{(1+u)^2} du =$$

$$2\int_0^\infty \frac{\cosh(u(1-q/p))}{(1+u)^2} du = +\infty, \quad p \neq q.$$

8. (10pt) Find and example of a sequence (f_n) of measurable functions that converges a.e. to a function f but $\lim \int f_n \neq \int f$. Why does this not contradict LDCT?

   Solution. Exercises 7.A or 7.B provide examples. No contradiction arises because there is no integrable g that dominates f_n.

9. (10pt) Give examples of measure spaces $(X, \mathbf{X}, \mu)$ such that

   (a) $L^p \subsetneq L^q$, $1 \leq p < q \leq \infty$;

   (b) $L^p \subsetneq L^q$, $1 \leq q < p \leq \infty$;

   (c) $L^p \backslash L^q \neq \emptyset$, for any pair $1 \leq p \neq q \leq \infty$;

   (d) $L^p = L^q$, $1 \leq p \leq q \leq \infty$.

   Supply the necessary proofs.

   Comment. Since none of you chose this problem I'll save it for possible later use.