

BRIEF COMMUNICATIONS

On Harmonic Analysis of Causal Operators*

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ABSTRACT. We follow the group representation theory approach to define causal operators on Banach modules and present some of their spectral properties.

KEY WORDS: causal operator, representation of a locally compact Abelian group, Beurling spectrum, causally invertible operator, causal spectrum.

In this note, spectral representation theory for Abelian groups (spectral theory of Banach modules over group algebras) is used to define and study causal operators, which play an important role in system theory [1] and differential equations [2, 3]. We believe that the application of this theory offers some advantages over two well-known investigation methods in the theory of causal operators. The first of them pertains to operators acting in a Hilbert space, and the corresponding definition of causal operators uses a resolution of identity by a system of orthogonal projections [1]. The second approach, as presented, e.g., in [2,3], deals with causal operators in function spaces and is closely related to applications in the theory of functional equations, in particular, delay differential equations.

Let X_1 and X_2 be complex Banach spaces, let $\text{Hom}(X_1, X_2)$ be the Banach space of bounded linear operators from X_1 to X_2 , and let $\text{End } X = \text{Hom}(X, X)$ be the Banach algebra of endomorphisms of a Banach space X . Let \mathbb{G} be a locally compact Abelian group, and let $\widehat{\mathbb{G}}$ be the dual group of continuous unitary characters of \mathbb{G} . By $L_1(\mathbb{G})$ we denote the Banach algebra of complex functions integrable with respect to the Haar measure on \mathbb{G} with multiplication defined as the convolution of functions. If $f \in L_1(\mathbb{G})$, then by $\hat{f}: \widehat{\mathbb{G}} \rightarrow \mathbb{C}$ we denote the Fourier transform of f .

Each Banach space X considered here is assumed to be a Banach $L_1(\mathbb{G})$ -module with module structure defined by a strongly continuous isometric representation $T: \mathbb{G} \rightarrow \text{End } X$ according to the formula

$$T(f)x = fx = \int_{\mathbb{G}} T(-g)f(g)x \, dg, \quad f \in L_1(\mathbb{G}), \quad x \in X. \quad (1)$$

We shall often denote the $L_1(\mathbb{G})$ -module X by (X, T) and the operator $x \mapsto fx: X \rightarrow X$, $f \in L_1(\mathbb{G})$, by $T(f)$.

Definition 1 [4–6]. The *Beurling spectrum* of a vector x in an $L_1(\mathbb{G})$ -module (X, T) is the subset

$$\Lambda(x) = \Lambda(x, T) = \{\gamma \in \widehat{\mathbb{G}} : fx \neq 0 \text{ for each function } f \in L_1(\mathbb{G}) \text{ such that } \hat{f}(\gamma) \neq 0\}$$

of the dual group $\widehat{\mathbb{G}}$.

Example 1. Let H be a (complex) Hilbert space, and let $E: \Sigma \rightarrow \text{End } H$ be a strongly continuous bounded projection-valued measure on the σ -algebra Σ of Borel subsets of $\widehat{\mathbb{G}}$. Then the formula $T(g)x = \int_{\widehat{\mathbb{G}}}(\gamma, g) \, dE(\gamma)x$, $x \in H$, defines an isometric representation $T: \mathbb{G} \rightarrow \text{End } H$, and formula (1) gives the corresponding $L_1(\mathbb{G})$ -module structure on H . The set $\Lambda(x) = \Lambda(x, T)$ coincides with the support of E .

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Example 2. Let Δ be a measurable subset of nonzero measure in $\widehat{\mathbb{G}}$, and let $L_p(\Delta, Y)$, $p \in [1, \infty)$, be the Banach space of equivalence classes of (Bochner) measurable p -integrable functions $x(\gamma)$ on \mathbb{G} ranging in a Banach space Y ; the norm in $L_p(\Delta, Y)$ is given by $\|x\|_p = (\int \|x(\gamma)\|^p d\gamma)^{1/p}$. The $L_1(\mathbb{G})$ -module structure on $L_p(\Delta, Y)$ is defined by the formula $(V(f)x)(\gamma) = (fx)(\gamma) = \hat{f}(\gamma)x(\gamma)$, $f \in L_1(\mathbb{G})$, $x \in L_p(\Delta, Y)$, $\gamma \in \Delta$. The associated representation $V: \mathbb{G} \rightarrow \text{End } L_p(\Delta, Y)$ has the form $(V(g)x)(\gamma) = \gamma(g)x(\gamma)$, $g \in \mathbb{G}$, $\gamma \in \Delta$, and $\Lambda(x, V) = \text{supp } x$ for $x \in L_p(\Delta, Y)$.

Let $T_i: \mathbb{G} \rightarrow \text{End } X_i$, $i = 1, 2$, be two strongly continuous isometric representations. A special role is played in this paper by two Banach module structures that can be introduced on the operator space $\mathfrak{U} = \text{Hom}(X_1, X_2)$.

They are defined by formula (1) via the representations $\widetilde{T}: \mathbb{G} \times \mathbb{G} \rightarrow \text{End } \mathfrak{U}$, $\widetilde{T}(g_1, g_2)A = T_2(g_2)AT_1(g_1)$, and $T_0: \mathbb{G} \rightarrow \text{End } \mathfrak{U}$, $T_0(g)A = T_2(g)AT_1(-g)$, where $g, g_1, g_2 \in \mathbb{G}$ and $A \in \mathfrak{U}$.

The set

$$\mathfrak{U}_0 = \{A \in \mathfrak{U} : \text{the function } g \rightarrow T_0(g)A: \mathbb{G} \rightarrow \mathfrak{U} \text{ is norm continuous}\}$$

is a closed submodule in \mathfrak{U} and is simultaneously a subalgebra in \mathfrak{U} if $X_1 = X_2 = X$ and $T_1 = T_2 = T$.

Theorem 1 [6]. *The Beurling spectrum $\Lambda(A, T_0)$ of any linear operator $A \in \text{Hom}(X_1, X_2)$ can be represented in the form $\Lambda(A, T_0) = \overline{\{\gamma_2 - \gamma_1 : (\gamma_1, \gamma_2) \in \Lambda(A, \widetilde{T})\}}$.*

Corollary 1. $\Lambda(Ax, T_2) \subseteq \overline{\Lambda(A, T_0) + \Lambda(x, T_1)}$ for any $x \in X_1$ and $A \in \mathfrak{U}$.

Let (X, T) be a Banach $L_1(\mathbb{G})$ -module, and let σ be a closed subset in $\widehat{\mathbb{G}}$. Then the set $X(\sigma) = \{x \in X : \Lambda(x) \subseteq \sigma\}$, which is a closed submodule in X , is called a *spectral submodule*, or a *spectral subspace* [5, 6].

Let (X_i, T_i) , $i = 1, 2$, be Banach $L_1(\mathbb{G})$ -modules, and let $\mathbb{S} \subset \widehat{\mathbb{G}}$ be a closed semigroup such that 0 lies in the closure of the interior $\text{Int } \mathbb{S}$. A linear operator $A \in \mathfrak{U}$ is said to be *causal* with respect to \mathbb{S} (and T_0) if $\Lambda(Ax, T_2) \subseteq \Lambda(x, T_1) + \mathbb{S}$ for all $x \in X_1$. The set $\Lambda(A, T_0) \setminus \{0\} \subseteq \widehat{\mathbb{G}}$ will be called the *memory* of A (cf. [2]). It follows from Theorem 1 and Corollary 1 that a linear operator $A \in \mathfrak{U}$ is causal with respect to \mathbb{S} if and only if its memory is contained in \mathbb{S} . Thus the set of causal operators in \mathfrak{U} with respect to \mathbb{S} coincides with the spectral submodule $\mathfrak{U}(\mathbb{S})$, which will be denoted by $\mathfrak{Caus}(X_1, X_2)$. If $X_1 = X_2 = X$ and $T_1 = T_2 = T$, then $\mathfrak{Caus}(X, X)$ is a closed subalgebra in $(\text{End } X, T_0)$ and will be denoted by $\mathfrak{Caus}(X, T_0)$ or $\mathfrak{Caus}(X)$.

If $\mathbb{G} = \mathbb{R}$ ($\mathbb{G} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$), then for the semigroup $\mathbb{S} \subset \widehat{\mathbb{R}} = \mathbb{R}$ (respectively, $\mathbb{S} \subset \widehat{\mathbb{T}} = \mathbb{Z}$) one usually takes $\mathbb{S} = \mathbb{R}_+ = [0, \infty)$ (respectively, $\mathbb{S} = \mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$). It follows from Theorem 1 that for $\mathbb{G} = \mathbb{R}$ an operator $A \in \mathfrak{U}$ is causal with respect to \mathbb{R}_+ if and only if $AX_1([t, \infty)) \subseteq X_2([t, \infty))$, $t \in \mathbb{R}$ (see [1–3] and Example 1).

Example 3. Let $\mathbb{G} = \mathbb{T}$ and $\Delta \subseteq \mathbb{Z}$. Consider the algebra $\text{End } l_p$, where $l_p = l_p(\Delta) = L_p(\Delta, \mathbb{C})$, $p \in [1, \infty)$, is the Banach space of complex sequences defined on Δ . Consider the representation $T = V: \mathbb{T} \rightarrow \text{End } l_p$ in Example 2 and the corresponding representations $\widetilde{T}: \mathbb{T} \times \mathbb{T} \rightarrow \text{End}(\text{End } l_p)$ and $T_0: \mathbb{T} \rightarrow \text{End}(\text{End } l_p)$. Then $\Lambda(A, \widetilde{T}) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : a_{ij} \neq 0\}$ for each $A \in \text{End } l_p$, where $\mathcal{A} = (a_{ij})$, $i, j \in \Delta$, is the matrix of A with respect to the standard basis in l_p ; furthermore, the set $\Lambda(A, T_0)$ consists of those $k \in \Delta - \Delta$ for which there exist $i, j \in \Delta$ such that $i - j = k$ and $a_{ij} \neq 0$. Therefore, an operator A is causal with respect to \mathbb{Z}_+ if and only if its matrix $\mathcal{A} = (a_{ij})$ is lower triangular.

Let $\gamma_0 \in \widehat{\mathbb{G}}$. A bounded net (f_α) in $L_1(\mathbb{G})$ is called a γ_0 -net if $\hat{f}_\alpha(\gamma_0) = 1$ for all α and $\lim_\alpha f_\alpha * f = 0$ for all $f \in L_1(\mathbb{G})$ such that $\hat{f}(\gamma_0) = 0$. A point $\gamma_0 \in \Lambda(x, T) \subseteq \widehat{\mathbb{G}}$ is said to be *ergodic* for a vector x in the $L_1(\mathbb{G})$ -module (X, T) if there exists a limit $x_0 = \lim_\alpha f_\alpha x$ for some γ_0 -net (f_α) . The set of ergodic points will be denoted by $\Lambda_{\text{erg}}(x, T)$. If $x_0 = 0$, then γ_0 will be called a point of the *continuous Beurling spectrum* of x . If $x_0 \neq 0$, then x_0 is an eigenvector of X , i.e., satisfies $T(g)x_0 = \gamma_0(g)x_0$, $g \in \mathbb{G}$; in particular, $fx_0 = \hat{f}(\gamma_0)x_0$, $f \in L_1(\mathbb{G})$, whence it

follows that $\Lambda(x_0) = \{\gamma_0\}$. In this case, the character γ_0 will be called an *eigencharacter*. The set of eigencharacters will be called the *Bohr spectrum* of x and denoted by $\Lambda_B(x, T)$.

Now consider the $L_1(\mathbb{G})$ -module $\mathfrak{U} = (\text{Hom}(X_1, X_2), T_0)$. If $\Lambda(A_0, T_0) = \{0\}$ for $A_0 \in \mathfrak{U}$, then $T_2(g)A_0 = A_0T_1(g)$, $g \in \mathbb{G}$, and one says that A_0 is *memoryless*. The set of memoryless operators will be denoted by $\mathcal{M} = \mathcal{M}(\mathfrak{U})$. If $0 \in \Lambda_{\text{erg}}(A, T_0)$, then the operator $A_0 = \lim f_\alpha A \in \mathcal{M}$ (for some 0-net f_α) will often be denoted by $M(A)$. If $M(A) = 0$, then A is said to be *uniformly causal*. The set of uniformly causal operators will be denoted by \mathcal{UC} . Note that the operators in Example 3 are memoryless if and only if they have diagonal matrices and uniformly casual if and only if their matrices are strictly lower triangular.

An operator $A \in \mathbf{Caus}(X_1, X_2)$ is said to be *causally invertible* if $A^{-1} \in \text{Hom}(X_2, X_1)$ is a causal operator with respect to the representation $T_0^{-1}: \mathbb{G} \rightarrow \text{End Hom}(X_2, X_1)$ defined by the formula $T_0^{-1}(g)B = T_1(g)BT_2(-g)$, $B \in \text{Hom}(X_2, X_1)$. If $X_1 = X_2 = X$, then the symbol $\sigma_{\mathbf{Caus}}(A)$ stands for the spectrum of A in $\mathbf{Caus}(X)$.

Note that $\sigma(A) \subset \sigma_{\mathbf{Caus}}(A)$ and the inclusion may be strict. For example, if A is the operator in $L_2(\mathbb{T})$ given by the formula $(A\varphi)(\gamma) = \gamma\varphi(\gamma)$, $\gamma \in \mathbb{T}$, then $\sigma(A) = \mathbb{T}$ but $\sigma_{\mathbf{Caus}}(A) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. (The module structure in $L_2(\mathbb{T})$ is determined by the group of translations in $L_2(\mathbb{T})$.)

Theorem 2. *Let $A \in \mathbf{Caus}(X_1, X_2)$ be a causally invertible operator, and let $0 \in \Lambda_{\text{erg}}(A, T_0)$. Then $0 \in \Lambda_{\text{erg}}(A^{-1}, T_0^{-1})$, the operator $M(A)$ is invertible, and $(M(A))^{-1} = M(A^{-1})$.*

Corollary 2. $\sigma(M(A)) = \sigma_{\mathbf{Caus}}(M(A)) \subseteq \sigma_{\mathbf{Caus}}(A)$.

We present several sufficient conditions for an operator A to belong to the radical $\mathfrak{Rad}_{\mathbf{C}}(X) = \{B \in \mathbf{Caus}(X) : \sigma(BC) = \{0\} \text{ for an arbitrary } C \in \mathbf{Caus}(X)\}$ of the algebra $\mathbf{Caus}(X)$. In this case, $X_1 = X_2 = X$ and $T_1 = T_2 = T$. Throughout the following, we assume that for any compact sets $K_1 \subset \mathbb{S} \setminus \{0\}$ and $K_2 \subset \mathbb{S}$ there exists an $m \in \mathbb{N}$ such that $mK_1 \cap K_2 = \emptyset$, where $mK_1 = \underbrace{K_1 + K_1 + \dots + K_1}_{m \text{ times}}$.

Theorem 3. *An operator $A \in \mathfrak{U}_0$ in the algebra $\mathbf{Caus}(X)$ belongs to the radical of this algebra if the following conditions hold: (1) $A \in \mathcal{UC}$; (2) for each $\varepsilon > 0$ there exists a function $\varphi \in L_1(\mathbb{G})$ such that $\|T(\varphi)A - A\| < \varepsilon$.*

Corollary 3. *If $A \in \mathcal{UC} \cap \mathfrak{U}_0$ is a compact operator, then $A \in \mathfrak{Rad}_{\mathbf{C}}(X)$.*

Theorem 4. *Let $A \in \mathbf{Caus}(X)$ be a compact operator, and let the Bohr spectrum $\Lambda_B(x)$ of each vector x in the Banach $L_1(\mathbb{G})$ -module (X, T) be empty. Then $A \in \mathfrak{Rad}_{\mathbf{C}}(X)$.*

Corollary 4. *Let $A, K \in \mathbf{Caus}(X, T_0)$, let K be a compact operator, and let $\Lambda_B(x)$ be empty for each vector x in the module (X, T) . Then $\sigma_{\mathbf{Caus}}(A + K) = \sigma_{\mathbf{Caus}}(A)$.*

By a *spectral component* of the spectrum $\sigma(a)$ of an element a in a Banach algebra \mathcal{B} we mean a nonempty subset $\sigma_1 \subset \sigma(a)$ such that $\text{dist}(\sigma_1, \sigma(a) \setminus \sigma_1) > 0$.

Theorem 5. *Let $A \in \mathbf{Caus}(X) \cap \mathfrak{U}_0$, and let $0 \in \Lambda_{\text{erg}}(A, T_0)$. Then each spectral component of $\sigma_{\mathbf{Caus}}(A)$ contains at least one spectral component of $\sigma(M(A))$. In particular, if $A \in \mathcal{UC}$, then $\sigma_{\mathbf{Caus}}(A)$ is connected and $0 \in \sigma_{\mathbf{Caus}}(A)$.*

Example 4. Let A be the causal operator in Example 3, where $\Delta = \mathbb{N}$. Hence its matrix $\mathcal{A} = (a_{ij})$ is lower triangular. It follows that $\sigma_{\mathbf{Caus}}(A)$ contains the set $\sigma_0 = \overline{\{a_{ii}; i \in \Delta\}}$ and each spectral component of $\sigma_{\mathbf{Caus}}(A)$ contains at least one of the numbers a_{ii} , $i \geq 1$. The relation $\sigma_{\mathbf{Caus}}(A) = \sigma(A)$ holds provided that the following two conditions are satisfied: (1) $\lim_{k \rightarrow \infty} a_{i, i+k} = 0$, $i \geq 1$; (2) $\sum_{k \geq 1} d_k(A) < \infty$, where $d_k(A) = \sup_{i \geq 1} |a_{i, i+k}|$.

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