M4111, TEST 1.

1. Prove that $(\bigcup_i X_i)^c = \bigcap_i X_i^c$.

   Solution. See pp. 6-7 of the textbook.

2. State the definition of a bounded set in a metric space. Prove that the set $\{a^{n/2},\ n \in \mathbb{N}\}$ is unbounded if $a > 1$.

   Solution. See the corresponding problem in the homework.

3. Mark all the statements each of which separately ensures that the subset S of a metric space (X, d) is closed.

   (a) S^c is open;
   (b) $\partial S \subseteq S$;
   (c) $\partial S \subseteq S^c$;
   (d) S contains all its cluster points;
   (e) $S^o \subseteq S$;
   (f) $S = \emptyset$;
   (g) $\bar{S} = S$;
   (h) $S \cap \overline{S^c} = \emptyset$;
   (i) $\bar{S} \cap S^c = \emptyset$;
   (j) $d(q, S) > 0$ for all $q \in S^c$;
   (k) $(S^c)^o$ is open;
   (l) $(S^c)^o$ is closed.

   Does any of your answers change if you assume in addition that the space X is connected?

   Solution. You should have marked: (a), (b), (d), (f), (g), (i), (j). Connectedness changes nothing unless you also assume that S is not dense. In that case $S = \emptyset$ in (l).

4. State and prove the **monotone convergence theorem**.

   Solution. See the text on p. 50.

5. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Recall the definitions of $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$. Show that $\limsup_{n\to\infty} a_n \geq \liminf_{n\to\infty} a_n$ and that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$ implies that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

Solution. This is problem 18 on p.63 of your textbook. The definitions are stated there. Here is how you should understand $\limsup$ and $\liminf$. Look at all convergent subsequences of your sequence and their limits. The largest of them is $\limsup$ and the smallest is $\liminf$. Clearly any bounded sequence of the real numbers has $\limsup$ and $\liminf$. Let $a > R = \limsup a_n$. Then $(a+R)/2 < a_n$ for only finitely many a_n and $(a+R)/2 > a_n$ for infinitely many a_n. Hence, a is not a lower bound for the set $\{x \in \mathbb{R} : a_n < x \text{ for an infinite number of integers } n\}$. Hence, $a > \liminf a_n = L$, and, since a is an arbitrary number bigger than R, we have $R \geq L$.

By definition, we have that only finitely many members of the sequence satisfy $a_n > R + \epsilon$ or $a_n < L - \epsilon$ for any $\epsilon > 0$. Thus, if $L = R$, we have $L = R = \lim a_n$ by definition of the limit.

6. Show that the sequence

$$1,\ 1+\ln 2,\ 1+\ln(1+\ln 2),\ 1+\ln(1+\ln(1+\ln 2)),\ \ldots$$

is convergent and find the limit.

Solution. First of all, since $\ln 2 > 0$, the sequence is well defined and bounded below by 1. Now let us look at the function $f(x) = 1+\ln x - x$. Clearly, $f(1) = 0$ and $f'(x) = \frac{1}{x} - 1$. Hence, f is increasing on $(0,1)$ and decreasing on $(1,\infty)$. This immediately tells us all we need to know:

(i) Starting from the second element our sequence is decreasing (because $f(x) \leq 0$). Hence $L = \lim a_n$ exists.

(ii) The only root of f is 1. Hence, $L = 1$.

7. Recall the space $\ell^\infty(\mathbb{N})$ from h/w 3. This is the space of bounded infinite sequences of real numbers with the metrics $d(x,y) = \sup\limits_{n\in\mathbb{N}} |x_n - y_n|$. Show that this is a complete metric space. Provide an example of a closed bounded set in $\ell^\infty(\mathbb{N})$ that is not compact (prove it).

Solution. In the h/w problem you've shown that ℓ^∞ is a metric space. Let $x^{(n)}$ be a Cauchy sequence in ℓ^∞. It is immediate that for all n the sequence $\{x_m^{(n)}\}$ is Cauchy in $\mathbb{R}$. But $\mathbb{R}$ is complete and, hence, all these sequences converge. Call the corresponding limits x_m and the resulting sequence x. Let's show that $x = \lim x^{(n)}$. Suppose not. Let $\epsilon > 0$ be such that for all $N \in \mathbb{N}$ there exists $n > N$ such that $d(x^{(n)}, x) > \epsilon$ and, hence, there exists $m \in \mathbb{N}$ such that

$|x_m^{(n)} - x_m| > \epsilon$. But $x_m = \lim_{n\to\infty} x_m^{(n)}$. Hence, there exists $N_1 \in \mathbb{N}$ such that $\epsilon/2 < |x_m^{(n)} - x_m^{(k)}| < d(x^{(n)}, x^{(k)})$ for all $k > N_1$. This contradicts the fact that $\{x_m^{(n)}\}$ is Cauchy. Finally, we need to show that $x \in \ell^\infty$. For $y \in \ell^\infty$ we introduce the norm $\|y\| = \sup\limits_{n\in\mathbb{N}} |y_n| < \infty$. Since $x^{(n)}$ is Cauchy, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_m^{(n)}| < \|x^{(N)}\| + \epsilon$ for all $n > N$ and $m \in \mathbb{N}$. Hence, $\|x\| < \infty$, which means that $x \in \ell^\infty$.