M4111, TEST 2.
October 27, 2005.

Instructions: Don't pay any attention to Problems 4-6 until you've done everything you could about Problems 1-3. Each of the first 4 problems is worth a maximum of 20 points. Each of the last 2 problems - 25 points. Thus, maximum you can get is 130. The "perfect" score is 100.

1. Here I am after the definitions.

   (a) Give an "inverse image" definition of a continuous function $f : E \to E'$.
   See Proposition on p. 70.

   (b) A sequence of functions $f_n : E \to E'$ is unifomly convergent if ...
   See Definition on p. 85.

   (c) A subset S of a metric space E is arcwise connected if ...
   See Problem 29 on p. 93.

   (d) A function $f : E \to E'$ is unifomly continuous on $S \subseteq E$ if ...
   See Definition on p. 80.

2. Now it's time for examples and counterexamples. If you think that the examples I ask for do not exist, you should prove it.

   (a) Give an example, if possible, of a function $f : (0,1) \to \mathbb{R}$ which is continuous and bounded but not uniformly continuous.
   $f(x) = \sin \frac{1}{x}$.

   (b) Give an example, if possible, of a continuous real-valued function whose image is not a connected set.
   $f : (0,1) \cup (2,3) \to \mathbb{R}$, $f(x) = x$.

   (c) Give an example, if possible, of a function $f : \mathbb{R} \to \mathbb{R}$ which has a uniformly continuous restriction to $\mathbb{Q}$ but is not uniformly continuous itself.
   The Dirichlet function.

3. And now you are to prove some theorems.

   (a) Prove that the image of a compact set under a continuous function is compact.

(b) Use the result above to deduce the existence of maximum and minimum values for real-valued continuous functions on compact sets (Extremal Value Theorem).

See pp. 78-79.

4. Here is something you surely know from calculus but never bothered to prove rigorously. Let $U \subseteq \mathbb{R}$ be an open interval, let $a \in U$, let E' be a metric space, and let $f : U \to E'$ be a function. Let also f_+ be the restriction of f to $U \cap [a, +\infty)$ and f_- be the restriction of f to $U \cap (-\infty, a]$. Define

$$\lim_{x \to a^+} f(x) = \lim_{x \to a} f_+(x) \quad \text{and} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a} f_-(x).$$

Show that $\lim_{x \to a} f(x)$ exists iff both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal.

Solution. The most common mistake was misinterpreting the definition of the limit. U must be a DELETED neighbourhood.

$(\lim_{x \to a} f(x) = L) \Leftrightarrow ((\forall \epsilon > 0)(\exists \delta > 0)(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon)) \Leftrightarrow (((\forall \epsilon > 0)(\exists \delta > 0)(0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon) \wedge ((\forall \epsilon > 0)(\exists \delta > 0)(0 < a - x < \delta \Rightarrow |f(x) - L| < \epsilon))) \Leftrightarrow ((\lim_{x \to a^+} f(x) = L) \wedge (\lim_{x \to a^-} f(x) = L))$.

5. Here is another problem from your text. Show that the limit of a uniformly convergent sequence of bounded functions from one metric space into another is a bounded function.

Solution. By uniform convergence, for some number $n \in \mathbb{N}$ we have that $\sup_x |f(x) - f_n(x)| \leq 1$. Assume that $\sup |f_n(x)| \leq M = M(n)$. Then, since, $-1 \leq (f(x) - f_n(x)) \leq 1$, we have $-1 - M \leq f(x) \leq 1 + M$.

6. And here is a problem from my text. Let $f : (a, b) \to \mathbb{R}$ be a continuous function and let $x_1, x_2, \ldots x_n \in (a, b)$ be arbitray points in the interval. Show that there exists $\xi \in (a, b)$ such that

$$f(\xi) = \frac{1}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)].$$

Solution. Denote $A = \frac{1}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)]$. Since $\min f(x_i) \leq A \leq \max f(x_i)$, and $f((a, b))$ is connected, $A \in f((a, b))$.