**Homework 10, Math 310, due 16th November 2009**

When proving a sequence $\{x_n\}$ is a CS, the steps must be always the same. Start with an $\epsilon > 0$, which you are not allowed to choose. Then you must say what $N \in \mathbb{N}$ you will choose. This could be a desription depending on $\epsilon$ and if it is not obvious such an $N$ exists, you must justify it. Then for this $N$, you must check that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$.

(1) Let $\{x_n\}$ be a sequence and define a new sequence $\{y_n\}$ by the formula, $y_k = 10^k$ for $1 \leq k \leq 100$ and $y_{100+k} = x_k$ for $k \in \mathbb{N}$. (So, $y_1 = 10, y_2 = 100, y_{101} = x_1, y_{102} = x_2$ etc.) Show that $\{x_n\}$ is a CS if and only if $\{y_n\}$ is a CS.

*Proof.* We first assume that $\{x_n\}$ is a CS. Let $\epsilon > 0$ be given. Then there exists an $N_1 \in \mathbb{N}$ so that for all $n, m \geq N_1$, $|x_n - x_m| < \epsilon$. Now, let $N = N_1 + 100$ and then for all $n \geq N$, we have, $y_n = x_{n-100}$. Thus for $n, m \geq N$, we have,

$$|y_n - y_m| = |x_{n-100} - x_{m-100}| < \epsilon,$$

since $n - 100, m - 100 \geq N - 100 = N_1$. This proves that $\{y_n\}$ is CS.

To prove the converse, assume that $\{y_n\}$ is a CS and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ so that for all $n, m \geq N$, we have $|y_n - y_m| < \epsilon$. Then if we choose the same $N$, for any $n, m \geq N$ we have,

$$|x_n - x_m| = |y_{n+100} - y_{m+100}| < \epsilon,$$

since $n + 100, m + 100 \geq N + 100 \geq N$. This finishes the proof. □

(2) Let $\{x_n\}$ be a sequence and define a new sequence $\{y_n\}$ by the formula, $y_k = x_{2k}$ for all $k \in \mathbb{N}$. Show that if $\{x_n\}$ is a CS so is $\{y_n\}$. Give an example to show that the converse may not hold.

*Proof.* Assume that $\{x_n\}$ is a CS and let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ so that for all $n, m \geq N$, $|x_n - x_m| < \epsilon$. For the same $N$, and $n, m \geq N$, we have,

$$|y_n - y_m| = |x_{2n} - x_{2m}| < \epsilon,$$

since $2n, 2m \geq 2N \geq N$. This proves that $\{y_n\}$ is a CS.

To show that the converse does not hold, let us consider the sequence $\{x_n\}$, where $x_n = (-1)^n$. Then the seqence $\{y_n\}$ has, $y_n = x_{2n} = (-1)^{2n} = 1$ for all $n$. Then for any $\epsilon > 0$, taking $N = 1$, we get that for any $n, m \geq 1$, $|y_n - y_m| = |1 - 1| = 0 < \epsilon$, proving that $\{y_n\}$ is a CS. To show that $\{x_n\}$ is not a CS, let us take $\epsilon = 1$. If an $N \in \mathbb{N}$ existed as required by the defintion of a CS, then we must have $|x_n - x_m| < 1$ for all $n, m \geq N$. But, we may choose $n, m \geq N$ so that $n$ is even and $m$ is odd. Then

$$|x_n - x_m| = |(-1)^n - (-1)^m| = |1 - (-1)| = 2,$$

and 2 is not less than 1. This shows that $\{x_n\}$ is not a CS. □

(3) Define a sequence $\{x_n\}$ of rational numbers as,

$$x_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

(a) Show by induction that $x_n = 1 - \frac{1}{n+1}$. (Hint: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.)

*Proof.* Proof is by induction on $n$. Since $x_1 = \frac{1}{2} = 1 - \frac{1}{2}$, we see that the statement is true for $n = 1$.
Assuming $x_n = 1 - \frac{1}{n+1}$, we show that $x_{n+1} = 1 - \frac{1}{n+2}$ to finish the induction. Observe that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Since $x_{n+1} = x_n + \frac{1}{(n+1)(n+2)}$, we get,

$$\begin{aligned} x_{n+1} =& 1 - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} \\ &= 1 - \frac{1}{n+2} \end{aligned}$$

Thus we have proved the result by induction. □

(b) Show that $\{x_n\}$ is a CS.

*Proof.* Assume that we are given an $\epsilon > 0$. Then choose $N \in \mathbb{N}$ so that $N > 2/\epsilon$. If $n, m \geq N$, we get

$$\begin{aligned} |x_n - x_m| &= \left| \left(1 - \frac{1}{n+1}\right) - \left(1 - \frac{1}{n+2}\right) \right| \\ &= \left| \frac{1}{n+2} - \frac{1}{n+1} \right| \\ &\leq \frac{1}{n+2} + \frac{1}{n+1} \\ &< \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \epsilon. \end{aligned}$$

This shows that the sequence is a CS. □

(4) Let $0 \leq b_n \leq a_n$ be rational numbers. Show that if the sequence defined as, $x_n = \sum_{k=1}^{n} a_k$ is a CS, so is the sequence defined as $y_n = \sum_{k=1}^{n} b_k$. (Comparison Test).

*Proof.* Given $\epsilon > 0$, by assumption there exists an $N \in \mathbb{N}$ so that $|x_n - x_m| < \epsilon$ for all $n, m \geq N$. But then we get for this $N$ and $n \geq m \geq N$,

$$\begin{aligned} |y_n - y_m| &= |\sum_{k=1}^{n} b_k - \sum_{k=1}^{m} b_k| \\ &= \sum_{k=m+1}^{n} b_k \leq \sum_{k=m+1}^{n} a_k \\ &= |x_n - x_m| < \epsilon. \end{aligned}$$

This proves that the sequence $\{y_n\}$ is a CS. □

(5) Show that the following sequences are CS.
(a) The sequence $\{z_n\}$ defined as,

$$z_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}.$$

*Proof.* Let $\{x_n\}$ be the sequence in problem 3. Then we have seen that if $N > 2/\epsilon$ where $\epsilon > 0$ is any positive rational number, then for $n, m \geq N$, $|x_n - x_m| < \epsilon$. So, if $n \geq m \geq N$, we get,

$$|x_n - x_m| = \sum_{k=m+1}^{n} \frac{1}{k(k+1)} < \epsilon. \tag{1}$$

So, to show that $\{z_n\}$ is a CS, let $\epsilon > 0$ be given and then choose $N > 4/\epsilon$. Let $n \geq m \geq N$. We look at two cases (though both are very similar). First case is when both

$n, m$ are odd or both even. In this case, we have,

$$\begin{aligned}
|z_n - z_m| &= \left|\frac{1}{m+1} - \frac{1}{m+2} + \cdots + \frac{1}{n-1} - \frac{1}{n}\right| \\
&= \left|\left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right| \\
&= \frac{1}{(m+1)(m+2)} + \frac{1}{(m+3)(m+4)} + \cdots + \frac{1}{(n-1)n} \\
&\leq \frac{1}{(m+1)(m+2)} + \frac{1}{(m+2)(m+3)} + \frac{1}{(m+3)(m+4)} + \\
&+ \cdots + \frac{1}{(n-1)n} \\
&< \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

by equation 1.

If one of $n, m$ is even and the other odd, we get,

$$\begin{aligned}
|z_n - z_m| &= \left|\frac{1}{m+1} - \frac{1}{m+2} + \cdots + \frac{1}{n-2} - \frac{1}{n-1} + \frac{1}{n}\right| \\
&= \left|\left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n-1}\right) + \frac{1}{n}\right| \\
&= \frac{1}{(m+1)(m+2)} + \frac{1}{(m+3)(m+4)} + \cdots + \\
&+ \frac{1}{(n-2)(n-1)} + \frac{1}{n} \\
&< \frac{1}{(m+1)(m+2)} + \frac{1}{(m+2)(m+3)} + \frac{1}{(m+3)(m+4)} + \\
&+ \cdots + \frac{1}{(n-2)(n-1)} + \frac{1}{n} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon
\end{aligned}$$

again by equation 1. Thus in either case, we have achieved the correct inequlaity, proving that the sequence is a CS. □

(b) The sequence $\{u_n\}$ defined as,

$$u_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

*Proof.* Again, let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ so that $N - 1 > \epsilon/2$. If $n \geq m \geq N$, then we have,

$$\begin{aligned} |u_n - u_m| &= \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots + \frac{1}{n^2} \\ &\leq \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)} + \cdots + \frac{1}{(n-1)n} \\ &< \epsilon \end{aligned}$$

by equation 1 again, proving that the sequence is a CS. □

(6) We consider the harmonic series, where

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

(a) Show that for any $n \in \mathbb{N}$,

$$\sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > \frac{1}{2}.$$

*Proof.*

$$\begin{aligned} \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} &> \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} \\ &= 2^n \frac{1}{2^{n+1}} \\ &= \frac{1}{2}. \end{aligned}$$

□

(b) Show that $\{x_n\}$ is not a CS.

*Proof.* To show that $\{x_n\}$ is not a CS, we may pick an $\epsilon$ and so let us pick it to be $1/2$. If an $N \in \mathbb{N}$ existed for this $\epsilon$ satisfying the conditions for a CS, we must have for all $n, m \geq N$, $|x_n - x_m| < 1/2$. Choose an $n$ so that $2^n \geq N$. This can be done by lemma 2.6 in the notes. Then clearly $2^{n+1} \geq N$ and so we must have $|x_{2^{n+1}} - x_{2^n}| < 1/2$. But,

$$|x_{2^{n+1}} - x_{2^n}| = \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} > \frac{1}{2}.$$

This contradiction proves that no such $N$ exists, showing that the sequence is not a CS. □