Homework 8, Math 310, due October 29th, 2012

(1) Prove properties 3,4 and 5 on page 8 of the notes. Remember that you are only allowed to use properties which are proved before (including 1 and 2).

(2) The following are applications of the universal property of natural numbers. Remember that to apply it, you need a set, a function from the set to itself and an element in the set. So, most of the problems would be done if you can find these three things appropriately and you are expected to state these three explicitly before you appeal to the universal property.

   (a) Let $x \in \mathbb{N}$. Show that there exists for each $n \in \mathbb{N}$ a natural number denoted by x^n (this is just a notation, but should tell you what we are doing) such that $x^1 = x$ and $x^{\sigma(n)} = x \cdot x^n$.

   (b) Let S be a set and let $f : S \to S$ be a function. Show that for any $n \in \mathbb{N}$, there exists a function denoted by $f^n : S \to S$ such that $f^1 = f$ and $f^{\sigma(n)} = f \circ f^n$. Further, show that $f \circ f^n = f^n \circ f$ for all $n \in \mathbb{N}$.

   (c) From the previous problem, we have $\sigma^n : \mathbb{N} \to \mathbb{N}$ for all $n \in \mathbb{N}$.

       (i) Show that for any $n \in \mathbb{N}$, $\sigma^{n+1}(\mathbb{N}) \subset \sigma^n(\mathbb{N})$, where we have used $n + 1$ for $\sigma(n)$ as we defined in class.

       (ii) Show that the set $\sigma^n(\mathbb{N}) - \sigma^{n+1}(\mathbb{N}) = \{\sigma(n)\}$.

       (iii) Let $\Sigma_n = \mathbb{N} - \sigma^n(\mathbb{N})$. Show that $\Sigma_{n+1} = \Sigma_n \cup \{n+1\}$ for all $n \in \mathbb{N}$.

Here is a problem in logical thinking. Do not submit and you may use anything you know.

Two mathematicians A, B meet on a street.

A: Haven't seen you in a long time. How are you? How many children do you have and how old are they?

B: I have three children and their ages (note: they are whole numbers) add up to the number of windows on that building across the street.

A: Oh, but I can't figure out their ages.

B: The product of their ages is 36.

A: Um! I am afraid I still don't know their ages.

B: Oh, did I tell you that the oldest plays the piano?

A: Now I know their ages.

What are their ages?