(1) Let \((X, T)\) be a topological space and let \(B \subset T\) with the following property. For any \(U \in T\) with \(x \in U\), there exists an element \(U_x \in B\) with \(x \in U_x\) such that \(U_x \subset U\). Show that any non-empty \(U \in T\) is the union of elements of \(B\) and thus \(T\) is the topology generated by \(B\). Such a set \(B\) is called a base for the topology \(T\).

(2) Let \(B \subset \mathcal{P}(\mathbb{R})\) be the set of all open intervals, \((a, b) = \{x \in \mathbb{R} | a < x < b\}\). Show that \(B\) is a base for the topology generated by \(B\) (which is the topology we have used for \(\mathbb{R}\)).

(3) In \(\text{Spec} \mathbb{Z}\) with the Zariski topology, show that any closed set which is not \(\text{Spec} \mathbb{Z}\) is a finite set.

(4) Let \((X, T_X), (Y, T_Y)\) be topological spaces and assume we give \(X \times Y\) the product topology. Show that the collection of the sets of the form \(U \times V\) where \(U \in T_X, V \in T_Y\) form a basis for the product topology. Show that if \(p_X : X \times Y \to X\) is the map \(p_X(x, y) = x\) (called the projection) then for any open set \(W\) in \(X \times Y, p_X(W)\) is open. (Such maps are called open maps).

(5) For a topological space \((X, T)\), show that it is Hausdorff if and only if the diagonal \(\Delta = \{(x, x) \in X \times X\}\) is closed in the product topology.

(6) (a) Show that the series \(\sum_{n=0}^{\infty} 10^{-n^2}\) converges. We will call the number that the series converges to by \(a\).

(b) We have seen that the topology induced on \(\mathbb{Z} \subset \mathbb{R}\) from the usual topology on \(\mathbb{R}\) is discrete. Show that the topology induced on the subset, \(\mathbb{Z} + \mathbb{Z}a = \{x \in \mathbb{R} | x = m + na, m, n \in \mathbb{Z}\}\) is not discrete.