Math 417, Homework 4, due 5th October 2010

(1) Let \( \mathbb{Q} \) be given the \( p \)-adic topology for a prime \( p \) and let \( U_n \) for an integer \( n \) as usual be the open set, \( U_n = \{ r \in \mathbb{Q} | v_p(r) \geq n \} \).

(a) Show that if \( r, s \in U_n \) then so is \( r \pm s \). (That is \( U_n \) is a subgroup of \( \mathbb{Q} \) with respect to addition).

(b) Show that \( U_n \) is closed. (Again, this is true for any topological group-any open subgroup is closed).

(c) Let \( S = \{ 1, p, p^2, \ldots \} \subset \mathbb{Q} \). Find all limit points (if any) of \( S \) in \( \mathbb{Q} \).

(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. Show that it is continuous if and only if for any sequence \( \{ x_n \} \) with \( x_n \in \mathbb{R} \) and \( \lim x_n = x \), one has \( \lim f(x_n) = f(x) \).

(3) We say that a subspace \( A \subset X \) is dense in \( X \) if the closure of \( A \) is \( X \). Show that if \( f, g : X \to Y \), where \( Y \) is Hausdorff, are two continuous functions and \( f|_A = g|_A \) (restrictions of \( f \) and \( g \) to \( A \)) where \( A \) is dense in \( X \), then \( f = g \).

(4) As usual, we identify the set of \( 2 \times 2 \) matrices \( M_2(\mathbb{R}) \) with \( \mathbb{R}^4 \) by the map, \( \phi : M_2(\mathbb{R}) \to \mathbb{R}^4 \),

\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = (a, b, c, d).
\]

Show that this is a bijection (trivial). Thus, we may define a topology on \( M_2(\mathbb{R}) \) by declaring that a subset \( U \subset M_2(\mathbb{R}) \) is open if and only if \( \phi(U) \) is open in \( \mathbb{R}^4 \).

(a) Show that the map \( M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to M_2(\mathbb{R}) \) given by matrix addition is continuous.

(b) Show that the map \( M_2(\mathbb{R}) \to \mathbb{R} \) given by \( A \mapsto \det A \) is continuous.