A quick survey of $p$-adic numbers

We fix a prime number $p$ and define a map $v_p : \mathbb{Q} - \{0\} \to \mathbb{Z}$ as follows. If $0 \neq r \in \mathbb{Q}$, we may write $r = a/b$ with $a, b \in \mathbb{Z}$ and $a, b \neq 0$. Then there are unique non-negative integers $v_p(a), v_p(b)$ such that $p^{v_p(a)}$ divides $a$ but $p^{v_p(a)+1}$ does not divide $a$. Similarly for $b$. Define $v_p(r) = v_p(a) - v_p(b)$. (Check that if we write $r = a/b = c/d$ with $a, b, c, d \in \mathbb{Z}$, then $v_p(a) - v_p(b) = v_p(c) - v_p(d)$ and thus $v_p(r)$ is well defined.)

We extend this to all of $\mathbb{Q}$ by declaring $v_p(0) = +\infty$. Here is an easy lemma.

Lemma 1.  \( (1) \) For any $r \in \mathbb{Q}$, $v_p(r) = v_p(-r)
(2) \) If $r, s \in \mathbb{Q}$ then $v_p(rs) = v_p(r) + v_p(s)
(3) \) For $0 \neq r \in \mathbb{Q}$, $v_p(r) \geq n$ for some $n$ if and only if we can write $r = p^na/b$ with $a, b \in \mathbb{Z}$ and $p$ does not divide $b$.
(4) \) $v_p(r + s) \geq \min\{v_p(r), v_p(s)\}$.

Define for any integer $n$, $A_n = \{ r \in \mathbb{Q} | v_p(r) \geq n \}$ and for any $x \in \mathbb{Q}$, $A_n + x = \{ a + x \mid a \in A_n \}$.

Lemma 2.  \( (1) \) If $a \in A_n$, so is $-a$.
(2) \) If $x, y \in A_n$, then so is $x \pm y$. (This follows from the last property in the above lemma [1].)
(3) \) $A_n \subset A_m$ if $n \geq m$.
(4) \) $A_n + x = A_n + z$ for any $z \in A_n + x$.
(5) \) $(A_n + x) \cap (A_m + y) = \emptyset$ or equal to $A_k + z$ where $k = \max\{m, n\}$ for some $z$.

Now we are ready to define the $p$-adic topology on $\mathbb{Q}$.

Lemma 3. Let $\mathcal{T}$ be the subset of the power set of $\mathbb{Q}$ containing the empty set and arbitrary unions of the form $A_n + x$ for varying $n \in \mathbb{Z}$ and $x \in \mathbb{Q}$. Then $\mathcal{T}$ is the topology generated by the collection $\{A_n + x\}$.

Proof. Since $A_n + x \in \mathcal{T}$, and any set in $\mathcal{T}$ is either empty or unions of $A_n + x$, we see that if we show $\mathcal{T}$ is a topology, we would have proved the lemma.

By choice $\emptyset \in \mathcal{T}$. Since $\mathbb{Q} = \bigcup_{x \in \mathbb{Q}} A_0 + x$, we see that $\mathbb{Q} \in \mathcal{T}$.

It is clear that union of elements of $\mathcal{T}$ is in $\mathcal{T}$.

Finally, if $U_1, \ldots, U_m \in \mathcal{T}$, we wish to show that $\cap U_i \in \mathcal{T}$. If any of the $U_i = \emptyset$, then so is the intersection and we are done. So, let us assume that $U_i \neq \emptyset$ for all $i$. Then we can write $U_i = (A_n + x_i)$. Thus, $\cap U_i = \bigcap (A_{n_1} + x_1) \cap \cdots \cap (A_{n_m} + x_m)$. If any of the intersections are empty, we may ignore those. If one of those is not empty, by
lemma above, we have \((A_{n_1} + x_1) \cup \cdots (A_{n_m} + x_m) = A_k + z\) and thus \(\cap U_i \in \mathcal{T}\). \(\square\)

Finally we show that \(\mathbb{Q}\) with the \(p\)-adic topology is Hausdorff. Let \(x \neq y \in \mathbb{Q}\). One has \(v_p(x - y) = n \in \mathbb{Z}\) (so \(n \neq \infty\)). I claim that \(A_{n+1} + x \cap A_{n+1} + y = \emptyset\), which will prove that \((\mathbb{Q}, \mathcal{T})\) is Hausdorff. If not, we have \(a + x = b + y\) for some \(a, b \in A_{n+1}\). So \(b - a = x - y\) and thus \(v_p(b - a) = v_p(x - y) = n\). But, \(a, b \in A_{n+1}\) implies by earlier lemma, \(b - a \in A_{n+1}\) and thus \(v_p(b - a) \geq n+1\). This is a contradiction.

**Zariski Topology on \(\text{Spec} \mathbb{Z}\)**

Let \(\text{Spec} \mathbb{Z} = \{0, 2, 3, 5, \ldots, p, \ldots\}\), where \(p\) stands for a prime number. For any \(0 \neq n \in \mathbb{Z}\), we define

\[
\text{Spec} \mathbb{Z}_n = \{a \in \text{Spec} \mathbb{Z} | a \text{ does not divide } n\}
\]

(Some people define \(\text{Spec} \mathbb{Z}_0 = \emptyset\), assuming by convention that 0 does not divide 0). Let \(\mathcal{T}\) be the set of all \(\text{Spec} \mathbb{Z}_n\) and the empty set. I claim that this is a topology on \(\text{Spec} \mathbb{Z}\). This follows from the following easy lemma.

**Lemma 4.**

(1) \(\text{Spec} \mathbb{Z}_n \cup \text{Spec} \mathbb{Z}_m = \text{Spec} \mathbb{Z}_{d}\) where \(d = \gcd(n, m)\).

(2) \(\text{Spec} \mathbb{Z}_n \cap \text{Spec} \mathbb{Z}_m = \text{Spec} \mathbb{Z}_{nm}\).

**Proof.** For the first, let \(a \in \text{Spec} \mathbb{Z}_n \cup \text{Spec} \mathbb{Z}_m\). Then \(a \in \text{Spec} \mathbb{Z}_n\) or \(a \in \text{Spec} \mathbb{Z}_m\), say \(a \in \text{Spec} \mathbb{Z}_n\). If \(a = 0\), then clearly \(a \in \text{Spec} \mathbb{Z}_d\). If \(a \neq 0\), then \(a\) is a prime not dividing \(n\) and since \(d\) divides \(n\), \(a\) can not divide \(d\). Thus \(a \in \text{Spec} \mathbb{Z}_d\). In the opposite direction, if \(a \in \text{Spec} \mathbb{Z}_d\), as before if \(a = 0\), then \(a \in \text{Spec} \mathbb{Z}_n \cup \text{Spec} \mathbb{Z}_m\). If \(a \neq 0\), then \(a\) is a prime not dividing \(d\). By the property of greatest common divisor, then \(a\) can not divide at least of \(n, m\) and then \(a \in \text{Spec} \mathbb{Z}_n\) or \(a \in \text{Spec} \mathbb{Z}_m\).

The second part is equally easy. \(\square\)

We have seen in class that this topology is not Hausdorff. One way of seeing this is to note that any non-empty subset in \(\mathcal{T}\) contains 0. If \(p\) is a prime number, for the topology to be Hausdorff, we must have open sets \(U_0, U_p\) which are neighbourhoods of 0, \(p\) respectively and whose intersection is empty. But, \(0 \in U_0\), being a neighbourhood of 0 and since \(p \in U_p\), \(U_p \neq \emptyset\) and thus contains 0. So \(0 \in U_0 \cap U_p\).