De Rham Cohomology

1. Definition of De Rham Cohomology

Let $X$ be an open subset of the plane. If we denote by $C^0(X)$ the set of smooth (i.e., infinitely differentiable functions) on $X$ and $C^1(X)$, the smooth 1-forms on $X$ (i.e., expressions of the form $f \, dx + g \, dy$ where $f, g \in C^0(X)$), we have natural differentiation map $d : C^0(X) \to C^1(X)$ given by

$$f \mapsto \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy,$$

usually denoted by $df$. The kernel for this map (i.e., set of $f$ with $df = 0$) is called the zeroth De Rham Cohomology of $X$ and denoted by $H^0(X)$. It is clear that these are precisely the set of locally constant functions on $X$ and it is a vector space over $\mathbb{R}$, whose dimension is precisely the number of connected components of $X$. The image of $d$ is called the set of exact forms on $X$. The set of $pdx + qdy \in C^1(X)$ such that $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$ are called closed forms. It is clear that exact forms and closed forms are vector spaces and any exact form is a closed form. The quotient vector space of closed forms modulo exact forms is called the first De Rham Cohomology and denoted by $H^1(X)$.

A path for this discussion would mean piecewise smooth. That is, if $\gamma : I \to X$ is a path (a continuous map), there exists a subdivision, $0 = t_0 < t_1 < \cdots < t_n = 1$ and $\gamma(t)$ is continuously differentiable in the open intervals $(t_i, t_{i+1})$ for all $i$. Given a form $\omega$ and a path $\gamma$, we can integrate the form along the path.

**Lemma 1.** If $\gamma(0) = P, \gamma(1) = Q$ and $\omega = df$, by fundamental theorem of calculus, we see that $\int_\gamma \omega = f(Q) - f(P)$.

If $\gamma$ is a closed path, we may think of $\gamma$ as a map from $I$ or $S^1$, whichever is convenient. Here is a self-evident lemma.

**Lemma 2.** If $\gamma : S^1 \to \mathbb{R}^2$ is a closed path, then $Y = \mathbb{R}^2 - \gamma(S^1)$ has a unique unbounded connected component.

**Proof.** Since $\gamma(S^1)$ is compact and hence bounded, we can find a closed bounded disc $D$ containing $\gamma(S^1)$. It is immediate that $\mathbb{R}^2 - D$ is a connected open set contained in $Y$ and hence contained in a connected component of $Y$. Any other connected component of $Y$ must be hence completely contained in $D$ and hence bounded. \qed

The union of the bounded connected components of $Y$ as above is called the open region inside the closed curve $\gamma(S^1)$ and the complement of the unbounded component in $\mathbb{R}^2$ is called the closed region inside the closed curve $\gamma(S^1)$.

**Lemma 3.** Let $\omega$ be a closed form on $X$. Then it is exact if and only if $\int_\gamma \omega = 0$ for all closed paths $\gamma$ in $X$.

**Proof.** If $\omega$ is exact, by lemma 1, we see that $\int_\gamma \omega = 0$. Conversely, given the vanishing, define a function on $X$ by the following formula. Clearly we may assume that $X$ is connected (and hence path connected). Fixing a point $a \in X$, for any $x \in X$, take a path $\gamma$ from $a$ to $x$ and define $f(x) = \int_\gamma \omega$. The vanishing implies that $f(x)$ does not depend on the path $\gamma$ and it is clear that $df = \omega$. \qed
2. Coboundary Homomorphism

Lemma 4 (partition of unity). Let $X$ be covered by open sets $\{U_\alpha\}$. Then there exists a collection of smooth non-negative functions $\phi_\alpha : X \to \mathbb{R}$ such that $\text{Supp} \phi_\alpha \subset U_\alpha$, the supports are locally finite and $\sum \phi_\alpha = 1$.

Let $X = U \cup V$, union of two open sets. By partition of unity, we have $\phi_i, i = 1, 2$ such that $\text{Supp} \phi_1 \subset U$ and $\text{Supp} \phi_2 \subset V$, $\phi_i$ smooth on $X$ and $\phi_1 + \phi_2 = 1$. If $f$ is a smooth function on $U \cap V$, letting $f_1(x) = f(x)\phi_2(x)$ for $x \in U \cap V$ and $f_1(x) = 0$ for $x \in U - U \cap V$, we see that $f_1$ is smooth on $U$. Defining similarly, $f_2(x) = -\phi_1(x)f(x)$ for $x \in U \cap V$ and $f_2(x) = 0$ for $x \in V - U \cap V$, we see that $f_1 - f_2 = f$.

Now we define the coboundary map $H^0(U \cap V) \to H^1(X)$ as follows. Let $f \in H^0(U \cap V)$. Write $f = f_1 - f_2$ for smooth functions $f_i$ on $U$ as in the previous paragraph. Then $df_1 - df_2 = df = 0$, since $f$ is locally constant and thus the two forms $df_i$ patch together to get a form $\omega$ on $X$. Since it is locally exact, we see that $d\omega = 0$ and hence it is closed and thus defines an element in $H^1(X)$. Easy to check that this is well defined. So, we get,

$$(1) \quad \partial : H^0(U \cap V) \to H^1(X)$$

One can easily check that this map is a vector space homomorphism. That is, $\partial(f + g) = \partial(f) + \partial(g)$ and $\partial(af) = a\partial(f)$ for any real number $a$.

Lemma 5. $\partial(f) = 0$, if and only if $f = f_1 - f_2$, where $f_1 \in H^0(U), f_2 \in H^0(V)$. The class of a closed form $\omega$ is in the image of $\partial$ if and only if $\omega|_U, \omega|_V$ are exact.

Proof. If $f = f_1 - f_2$ with $f_i$ locally constant, we have $df_i = 0$ and hence $\partial(f) = 0$.

Conversely, if $\partial(f) = d\phi$ where $\phi$ is a smooth function on $X$ (which is what we mean by a class is zero in $H^1(X)$), writing $f = f_1 - f_2$ as before, we see that $df_1 = d\phi|_U$ and $df_2 = d\phi|_V$ and thus letting $g_i = f_i - \phi$, we see that $dg_i = 0$ and hence $g_1 \in H^0(U), g_2 \in H^0(V)$ and $g_1 - g_2 = f$.

We have seen that if $\omega$ is in the image of $\partial$ then $\omega$ restricted to $U, V$ are exact by our definition. Conversely, if $\omega|_U = df_1, \omega|_V = df_2$, then letting $f = f_1 - f_2$, we have $df = 0$ and hence $f \in H^0(U \cap V)$ and $\partial(f) = \omega$. \hfill $\square$

3. Some computations

Lemma 6. Let $X$ be any of the following:

1. $\mathbb{R}^2$.
2. Open half planes, like $x > a$ or open quadrants like $x > a, y > b$.
3. Open rectangle or disc.

Then $H^1(X) = 0$

Proof. If $\gamma$ is a closed path in $X$, then the region enclosed by $\gamma$ in $\mathbb{R}^2$ is completely contained in $X$ and apply Green’s theorem. \hfill $\square$

Let $P = (x_0, y_0) \in \mathbb{R}^2$ and consider the form,

$$\omega_P = \frac{-(y - y_0)dx + (x - x_0)dx}{(x-x_0)^2 + (y-y_0)^2}.$$

Then $\omega_P$ is a smooth form everywhere except at $P$ and it is closed. Letting $X = \mathbb{R}^2 - \{P\}$, we see that for any circle $C$ around $P$, $\int_C \omega_P = 2\pi \neq 0$. Thus, by lemma
3, we see that \( \omega_P \neq 0 \) in \( H^1(X) \). If \( \omega \) is any other closed form on \( X \), let \( a = \int_C \omega \), and then letting \( \omega' = \omega - \frac{1}{2\pi} \omega_P \), we have, \( \omega' \) is a closed form with \( \int_C \omega' = 0 \). I claim, then that \( \omega' \) is exact.

So, let \( \omega \) be a closed form on \( X \) with \( \int_C \omega = 0 \). We wish to show that \( \omega \) is exact. For ease of notation, let us assume that \( \tilde{P} \) is the origin. Then \( X \) is covered by the four open sets,

\[
U_1 = \{x > 0\}, U_2 = \{y > 0\}, U_3 = \{x < 0\}, U_4 = \{y < 0\}.
\]

By lemma 6, \( \omega = df_i \) on \( U_i \). Thus, \( df_1 - df_2 = 0 \) in \( U_1 \cap U_2 \), which is connected and hence we see that \( f_2 = f_1 + c \) for some constant \( c \). Since \( df_2 = df_2 - c \), it is clear that we may replace \( f_2 \) by \( f_2 - c \) and hence assume that \( f_2 = f_1 \) in \( U_1 \cap U_2 \). Continuing, we may assume \( f_3 = f_2 \) in \( U_2 \cap U_3 \) and \( f_4 = f_3 \) in \( U_3 \cap U_4 \). Then we get, \( f_4 = f_1 + c \) in \( U_4 \cap U_1 \) for some constant \( c \).

Now cutting up our circle to be paths contained in \( U_i \)'s and calculating the integral of \( \omega \) with these \( f_i \)'s, we see that \( \int_C \omega = c \), which we have assumed to be zero. So, \( f_4 = f_1 \) in \( U_4 \cap U_1 \) and thus these \( f_i \)'s patch up to get a smooth function \( \phi \) on \( X \) and \( d\phi = \omega \). Thus \( \omega \) is zero in \( H^1(X) \).

This shows that \( H^1(X) \) is a one-dimensional vector space generated by the class of \( \omega_P \).

A similar argument will show that for any \( P \neq Q \in \mathbb{R}^2 \), \( H^1(\mathbb{R}^2 - \{P,Q\}) \) is a two dimensional vector space generated by \( \omega_P, \omega_Q \).

The form \( \omega_P \) and its integral is closely connected to winding numbers. Again, for convenience let us assume that \( P \) is the origin. If \( \gamma : I \to \mathbb{R}^2 - \{0\} \) is a (smooth) path, we have defined the winding number \( W(\gamma,0) \) as follows. We can subdivide the plane into small regions of the form \( a \leq \theta \leq b \) where \( b - a < 2\pi \) and then we can divide \( I \) as \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) so that \( \gamma([t_i, t_{i+1}]) \) is completely contained in these chosen regions. Then the angle \( \theta_i \) from \( \gamma(t_i) \) to \( \gamma(t_{i+1}) \) is well defined and we define \( W(\gamma,0) \) to be the sum of these \( \theta_i \)'s. (Actually, we defined it by dividing this number by \( 2\pi \).)

One consequence is,

**Lemma 7.** If \( \gamma \) is a path as above, then

\[
\int_\gamma \omega_0 = W(\gamma,0).
\]

One immediately has the following corollary.

**Corollary 8.** Let \( A \subset \mathbb{R}^2 \) be a closed connected set and let \( P,Q \in A \). Then the class of \( \omega_P, \omega_Q \) are same in \( H^1(\mathbb{R}^2 - A) \).

**Proof.** Let \( \gamma \) be a closed path in \( \mathbb{R}^2 - A \). Suffices to show that \( \int_\gamma \omega_P = \int_\gamma \omega_Q \) by lemma 3. From the lemma above, suffices to show that \( W(\gamma,P) = W(\gamma,Q) \). Reversing the roles, \( W(\gamma,x) \) is a locally constant function on \( \mathbb{R}^2 - \gamma \) and since \( P,Q \) are in the same connected component of this set, since \( A \) is connected, we see that \( W(\gamma,P) = W(\gamma,Q) \).

4. Important Consequences

**Theorem 9.** Let \( \phi : I \to \mathbb{R}^2 \) be a homeomorphism to the image. Then \( \mathbb{R}^2 - \phi(I) \) is connected.

**Proof.** Let \( Y = \phi(I) \) and assume that the complement is not connected. Fix points \( P,Q \) in different connected components of \( \mathbb{R}^2 - Y \). Let \( A = \phi([0,1/2]) \)
and \( B = \phi([1/2,1]) \) and let \( S = \phi(1/2) \). Let \( U = \mathbb{R}^2 - A, V = \mathbb{R}^2 - B \). Then \( U \cap V = \mathbb{R}^2 - Y \) and \( U \cup V = \mathbb{R}^2 - \{S\} \). We have the coboundary homomorphism,

\[ \partial : H^0(U \cap V) \to H^1(U \cup V). \]

Since the \( H^1 \) is a one dimensional vector space generated by \( \omega_S \), for any \( f \in H^0(U \cap V) \), \( \partial(f) = a\omega_S \) for some \( a \in \mathbb{R} \). By lemma 5, this means that \( a\omega_S \) is exact on \( U, V \). Any circle \( C \) of large radius around \( S \) is contained in both \( U, V \). By lemma 3, we must have \( \int_C a\omega_S = 0 \), but this is just \( 2\pi a \). So, \( a = 0 \). In other words, the image of \( \partial \) is zero.

Pick a locally constant function \( f \) on \( U \cap V = \mathbb{R}^2 - Y \) such that \( f(P) \neq f(Q) \), which is possible, since \( P, Q \) are in different connected components. \( \partial(f) = 0 \) implies by lemma 5 that there exists \( f_1 \in H^0(U), f_2 \in H^0(V) \) such that \( f_1 - f_2 = f \). But, then either \( f_1(P) \neq f_2(Q) \) or \( f_2(P) \neq f_1(Q) \). Since \( f_i \)'s are locally constant, this means \( P, Q \) are in different connected components of \( U \) or \( V \). Fixing one such, we see that \( P, Q \) are in different connected components of say \( \mathbb{R}^2 - A \). Now call \( A = Y_1 \) and repeat the argument.

So, we get a sequence of closed intervals, \( I \supset I_1 \supset I_2 \supset \cdots \) with length of \( I_n = 2^{-n} \) and \( P, Q \) are in different connected components of \( \mathbb{R}^2 - Y_n \), where \( Y_n = \phi(I_n) \). By nested interval theorem, \( \cap_{n=1}^\infty Y_n = \{T\} \). But \( \mathbb{R}^2 - \{T\} \) is connected and so we can find a path connecting \( P, Q \) in this open set. So, there exists a small disc around \( T \) which does not intersect this path. It is immediate that \( Y_n \) for large \( n \) must be contained in this disc. So, the path does not intersect \( Y_n \) for large \( n \) and thus \( P, Q \) are in the same connected component of \( \mathbb{R}^2 - Y_n \) for large \( n \), contradicting our earlier assertion. This proves the theorem.

\[ \square \]

**Theorem 10** (Jordan Curve Theorem). Let \( \phi : S^1 \to \mathbb{R}^2 \) be a homeomorphism onto its image. Then \( \mathbb{R}^2 - \phi(S^1) \) has exactly two connected components, one unbounded and the other bounded.

**Proof.** The second part will follow from what we have already proved, if we prove the first part. Let \( Y = \phi(S^1) \) and let \( P \neq Q \) two points on \( Y \). Then \( Y \) can be written as the union of two paths from \( P, Q \), both homeomorphic to the unit interval. Call these \( A, B \). Then \( Y = A \cup B \) and let \( U = \mathbb{R}^2 - A, V = \mathbb{R}^2 - B \). So, we have \( U \cap V = \mathbb{R}^2 - Y \) and \( U \cup V = \mathbb{R}^2 - \{P, Q\} \). We wish to show that \( H^0(U \cap V) \) is two dimensional.

From the previous theorem, we know that \( H^0(U), H^0(V) \) both are one-dimensional, consisting of the constant functions. If \( f \in H^0(U \cap V) \) with \( \partial(f) = 0 \) by lemma 5, we can write \( f = f_1 - f_2 \) with \( f_i \) both constant functions on \( U, V \) respectively. Then it is clear that this kernel is one dimensional. For \( f \in H^0(U \cap V) \), we can write \( \partial(f) = a\omega_P + b\omega_Q \) for \( a, b \in \mathbb{R} \). Again, by lemma 5, this form must be exact on \( U, V \). Taking a large circle \( C \) containing \( Y \), we see that,

\[ \int_C a\omega_P + b\omega_Q = 2\pi(a + b). \]

Since this must be zero, we see that \( a + b = 0 \). Thus the image of \( \partial \) is contained in the one dimensional vector space generated by \( \omega = \omega_P - \omega_Q \). We will show that this is in the image and then we will have \( H^0(U \cap V) \) to be two dimensional.

So, we want to show that \( \omega \) restricted to both \( U, V \) are exact. But by corollary 8, this is clear. This finishes the proof.

\[ \square \]