Math 418, Homework 10, due April 19th 2011

(1) Recall, for a continuous map $f : S^1 \to S^1$, we have the induced map $f_* : \pi_1(S^1) = \mathbb{Z} \to \pi_1(S^1) = \mathbb{Z}$, which is multiplication by an integer $d$, called the degree of $f$.

(a) Let $f, g$ be continuous maps from the circle to itself. Show that $\deg(f \circ g) = \deg f \cdot \deg g$ and $\deg fg = \deg f + \deg g$ where $fg(z) = f(z)g(z)$, the multiplication on the right as complex numbers with absolute value 1. (Hint: Consider the map $(f, g) : S^1 \times S^1 \to S^1, (f, g)(z, w) = f(z)g(w)$.)

(b) Show that, if $f, g$ are as above, then $f$ is homotopic to $g$ if and only if $\deg f = \deg g$.

(c) If $f(-P) = f(P)$ for all $P \in S^1$, show that $d$ is even.

(2) Prove that if $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^m$, then $m = 2$.

(3) If $X \subset Y \subset Z$ are topological spaces with $X$ a deformation retract of $Y$ and $Y$ a deformation retract of $Z$, show that $X$ is a deformation retract of $Z$.

(4) Let $X = SL_2(\mathbb{R})$ be the group of $2 \times 2$ matrices with determinant 1 over $\mathbb{R}$. Show that $\pi_1(X)$ is non-trivial. (Hint: Consider $Y \subset X$, the set of matrices of the form $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. Show that $Y$ is a retract of $X$).