HOMEWORK 10, DUE THU APR 15TH

All solutions should be with proofs, you may quote from the book or from previous home works

(1) Let A be a PID. A module D is called *divisible* if for any nonzero $a \in A$, the multiplication map $D \xrightarrow{a} D$ is onto.

(a) Show that K, the fraction field of A (which is naturally an A-module) is divisible. Also, if D is divisible, any quotient module of D is divisible.

(b) Let $N \subset M$ are modules and let $f : N \to D$ is a homomorphism, where D is divisible. Show that there is a homomorphism $g : M \to D$ such that $g(n) = f(n)$ for all $n \in N$. (Hint: You will need Zorn's lemma).

(c) If D is a divisible module and is a submodule of a module M, show that there is a submodule $N \subset M$ such that $N \oplus D \cong M$. (This means, $N + D = M, N \cap D = 0$).

(d) Let $M = A/pA$ where $p \in A$ is a prime. Show that M is the submodule of some divisible module.

(2) We consider the filed extension, $\mathbb{Q} \subset \mathbb{R}$.

(a) Show that $\sqrt{2}, \sqrt{3} \in \mathbb{R}$ are algebraic over $\mathbb{Q}$. Find a polynomial $P(X) \in \mathbb{Q}[X]$ of degree 4 such that $P(\sqrt{2} + \sqrt{3}) = 0$. Decide whether this polynomial is irreducible over $\mathbb{Q}$.

(b) Show that $\sqrt{2} + \sqrt[3]{5}$ is algebraic over $\mathbb{Q}$ of degree 6.

(3) We say an element in $a \in \mathbb{C}$ is an algebraic *integer*, if it satisfies an equation $a^n + a_1 a^{n-1} + \cdots + a_n = 0$ where $a_i \in \mathbb{Z}$. For example, $\sqrt{-1}, 2^{\frac{1}{5}}$ are algebraic integers.

(a) Show that if $a \in \mathbb{C}$ is algebraic over $\mathbb{Q}$, there is some positive integer N such that Na is an algebraic integer.

(b) If $a \in \mathbb{Q}$ is an algebraic integer, show that $a \in \mathbb{Z}$.

(c) If a is an algebraic integer, show that the ring $\mathbb{Z}[a]$ is a finitely generated module over $\mathbb{Z}$.

(d) Show that if a, b are algebraic integers, so are $a + b, ab$. (Do not attempt to find the polynomials satisfied by these.)

(4) Show that $\cos r\pi, \sin r\pi$ are algebraic, where $r \in \mathbb{Q}$ and the angles are in radians as usual. (De Moivre's theorem).

(5) Let F be a finite field with say q elements.

(a) Show that the characteristic of F is a prime number p and $q = p^m$ for some m.

(b) Show that $a^q = a$ for all $a \in F$.

(c) Let $F \subset L$ be a field extension and let $a \in L$ algebraic over F. Show that $a^{q^m} = a$ for some positive integer m.