HOMEWORK 13

All solutions should be with proofs, you may quote from the book or from previous home works

(1) Prove that a symmetric polynomial in $x_1, \dots, x_n$ is a polynomial in the elementary symmetric functions in $x_1, \dots, x_n$. (I do not know a short and elementary proof of this with the material we have covered).

(2) Let $\alpha_1, \dots, \alpha_n$ be the roots of $f(X) = X^n + a_1X^{n-1} + a_2X^{n-1} + \cdots + a_n$ and let $s_k = \sum_{i=1}^n \alpha_i^k$. Let $g(u) = u^n f(\frac{1}{u}) = 1 + a_1u + a_2u^2 + \cdots + a_nu^n$.
 (a) Show that $g'(u) = -g(u)(s_1 + s_2u + s_3u^2 + \cdots)$.

 (b) Prove *Newton's identities*, namely, $s_k + a_1s_{k-1} + a_2s_{k-1} + \cdots + a_{k-1}s_1 + ka_k = 0$ for $1 \leq k \leq n$ and,

 (c) $s_k + a_1s_{k-1} + \cdots + a_ns_{k-n} = 0$ for $k > n$.

(3) Let n be a positive integer and denote by $\omega = e^{2\pi i/n}$.
 (a) Show that a complex number z with $z^n = 1$ and for any $k, 1 \leq k < n$, $z^k \neq 1$ is of the form ω^r where $\gcd(r, n) = 1$. These are called *primitive* roots of unity and there are precisely $\phi(n)$ of them, where ϕ is the Euler function.

 (b) Show that $\mathbb{Q}(\omega)$ is the splitting field of $X^n - 1 \in \mathbb{Q}[X]$.

 (c) Show that the Galois group $G(\mathbb{Q}(\omega)/\mathbb{Q})$ is isomorphic to the group of units in the ring $\mathbb{Z}/n\mathbb{Z}$.

(4) If K is a field of characteristic zero and contains ω (as in the previous problem), show that the Galois group of the polynomial $X^n - a, a \in K$ is abelian.

(5) Let K be the splitting field over $\mathbb{Q}$ of $X^4 - 2x^2 - 1$. Calculate $G(K/\mathbb{Q})$ and find all fields F such that $\mathbb{Q} \subset F \subset K$.