HOMEWORK 2, DUE WED FEB 10TH

All solutions should be with proofs, you may quote from the book

(1) In the following problem, we are given a set and a suggested binary operation. Check whether it is indeed a binary operation and then check whether the set is a group with respect to this operation.
   (a) Let $\mathbb{R}/\mathbb{Z}$ denote the set of equivalence classes of $\mathbb{R}$ with the relation $a \sim b$ if $a - b \in \mathbb{Z}$ (which you checked is indeed an equivalence relation in the first homework). Define an operation by $[a] + [b] = [a + b]$

   (b) Let U be the set of 2×2 matrices of the form $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ for all $\theta \in \mathbb{R}$, with the operation being the usual matrix multiplication.

   (c) Let G, H be two groups and let the set be $G \times H$ and the binary operation is defined as $(g, h)(g', h') = (gg', hh')$.

(2) Let G be a group such that for any $a, b \in G$, $(ab)^2 = a^2b^2$. Show that G is abelian.

(3) Let G be a group and let $\{H_\alpha\}$ be a collection (possibly infinite) of subgroups of G.
   (a) Show that $\cap_\alpha H_\alpha$ is a subgroup of G.

   (b) Now, let the above collection be the set of *all* subgroups of G different from the trivial subgroup $\{e\}$. If $\cap_\alpha H_\alpha \neq \{e\}$, show that every element of G has finite order.

(4) Let G be a group and H a subgroup.
   (a) Show that for any $a \in G$, $aHa^{-1} = \{aha^{-1} | h \in H\}$ is a subgroup of G (called a *conjugate* of H).

(b) Let $N = \cap_{a \in G} aHa^{-1}$. Show that for any $x \in G$, $xNx^{-1} = N$.

(5) Let G be a group, H a subgroup of G of finite index.
(a) Show that the set $\{aHa^{-1} | a \in G\}$ is finite.

(b) Show that there is a subgroup $N \subset H$ of finite index such that $aNa^{-1} = N$ for all $a \in G$.

(6) Let G be an abelian group.
(a) If $a, b \in G$ with $o(a) = m, o(b) = n$. Show that G has an element c with $o(c) = \text{lcm}(m, n)$, the lowest common multiple of m, n.

(b) Assume G is finite. If the number of solutions in G to the equation $x^n = e$ is at most n for any positive integer n, show that G must be cyclic.