HOMEWORK 5, DUE THU MAR 4TH

All solutions should be with proofs, you may quote from the book or from previous home works

(1) Let G be a finite group and let p be the smallest prime dividing the order of G. Let H be a subgroup of G of index p. Show that H is normal.

(2) Let G be a group of order 231. Show that the 11-Sylow subgroup is in the center of G.

(3) Let G be a group of order p^2q, p, q primes. Show that either a p-Sylow subgroup or q-Sylow subgroup is normal.

(4) Let G be a group of order pq, $p < q$ primes.
   (a) If p does not divide $q-1$, show that G is cyclic.

   (b) If p divides $q-1$, show that there is a unique non-abelian group G up to isomorphism.

(5) Let $\mathbb{F}_p$ as usual denote the field of p elements (i. e. $\mathbb{Z}/p\mathbb{Z}$ for a prime p, where we have addition and multiplication as usual).
   (a) Calculate the order of $GL(n, \mathbb{F}_p)$.

   (b) Find a p-Sylow subgroup (more or less explicitly describe).

(6) Let G be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$ where p divides $o(G)$.
   (a) Prove that the p-Sylow subgroup of G is normal.

   (b) If P is the p-Sylow subgroup, then there exists a normal subgroup N such that $P \cap N = \{e\}$ and $PN = G$.