In the following, $k = \mathbb{C}$ and $G$ a finite group of order $n$. Unless otherwise mentioned, all vector spaces will be finite dimensional over $k$.

1. Let $V$ be a representation of $G$ (as always, this means that we are given a homomorphism of groups $\rho : G \to \text{Aut}_k(V)$). Show that $S^2(V), \wedge^2(V)$ are both naturally $G$-modules and show that

\[
\chi_{S^2V}(g) = \frac{1}{2} (\chi_V(g)^2 + \chi_V(g^2))
\]

\[
\chi_{\wedge^2 V}(g) = \frac{1}{2} (\chi_V(g)^2 - \chi_V(g^2))
\]

2. Let $G$ act on a finite set $S$ and let $V$ be the vector space with basis $S$. Then $V$ is a $G$-module. Show that $\chi_V(g)$ is the cardinality of $\{s \in S \mid gs = s\}$.

3. Let $V, W$ be two $G$-modules. Show that

\[
\chi_{\text{Hom}(V,W)}(g) = \overline{\chi_V(g)\chi_W(g)}.
\]

4. Let $G = S_3$ be the permutation group on three elements $S = \{s_1, s_2, s_3\}$. Then as we have seen the three dimensional vector space on $S$ is a $G$-module. Let $\sigma, \tau \in G$ with $\sigma^2 = e, \tau^3 = e$ and $\sigma \tau \sigma = \tau^{-1}$ as usual.

(a) Show that $G$ has two non-isomorphic one dimensional representations, one the trivial representation and the other given by $f : G \to \mathbb{C}^*$, where $f(\tau) = 1, f(\sigma) = -1$.

(b) Show that the 1-dimensional subspace of $V$ generated by $\sum s_i$ is a $G$-submodule.

(c) Show that the 2-dimensional $G$-module $W = V/\mathbb{C}(\sum s_i)$ is an irreducible representation of $G$. We will later see that these are all the irreducible representations of $G$ up to isomorphism.