CONSTRUCTION OF LOW RANK VECTOR BUNDLES ON $\mathbf{P}^4$ AND $\mathbf{P}^5$

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Abstract. We describe a technique which permits a uniform construction of a number of low rank bundles, both known and new. In characteristic two, we obtain rank two bundles on $\mathbf{P}^5$. In characteristic p, we obtain rank two bundles on $\mathbf{P}^4$ and rank three bundles on $\mathbf{P}^5$. In arbitrary characteristic, we obtain rank three bundles on $\mathbf{P}^4$ and rank two bundles on the quadric S_5 in $\mathbf{P}^6$.

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1. Introduction

The purpose of this paper is to give a technique which allows the construction of a number of non-split vector bundles of rank two and three on projective four space and five space. There is a relatively short list of such bundles known at this time. We mention in characteristic zero the Horrocks-Mumford bundle on $\mathbf{P}^4$ [4], Tango's rank three bundle on $\mathbf{P}^4$ [8] (known also to Trautmann [10]), Sasakura's rank three bundle on $\mathbf{P}^4$ [1], the family of rank three bundles on $\mathbf{P}^5$ given by Horrocks' parent

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bundle and its affine 'pull-backs'. Of course restrictions from $\mathbf{P}^5$ to $\mathbf{P}^4$, pull-backs by finite morphisms, extensions of rank two bundles by line bundles can be used to build more such bundles. In finite characteristic, there is Tango's rank two bundle on $\mathbf{P}^5$ in characteristic two [9], and also the rank two bundles on $\mathbf{P}^4$ created in any finite characteristic by Mohan Kumar [7], and also possible specializations of characteristic zero examples.

Our technique arose out of an exploration of the last cited result. It permits the construction of a number of low rank bundles: in finite characteristic, we obtain rank two bundles on $\mathbf{P}^5$ (in characteristic two), rank two bundles on $\mathbf{P}^4$, rank three bundles on $\mathbf{P}^5$. In arbitrary characteristic, our technique allows us to construct rank three bundles on $\mathbf{P}^4$ and a rank two bundle on the quadric S_5 in $\mathbf{P}^6$. For the most part, we do not create a roadmap of when our constructions coincide with known examples and when they are new bundles. Our methods do not reproduce bundles such as the Horrocks-Mumford bundle or Tango's rank three bundle even in their characteristic p specializations. However, we have some bundles which are new in the sense that they are not modifications of bundles from the known list.

The strength of the technique is to show that the same phenomenon underlies many low rank bundles. Our approach is to use Horrocks' ideas about extending to the ambient space the syzygies of a bundle on a divisor (see the description given in [3]). Our method essentially goes as follows: on a divisor of a smooth variety X, find a rank two bundle which is generated by four global sections. Then there is a surjection from four copies of $\mathcal{O}_X$ to the bundle. The kernel is a rank four bundle $\mathcal{G}$ on X. $\mathcal{G}$ can be viewed as an extension to X of the rank two bundle and its syzygy. We now exploit this $\mathcal{G}$ to find a line sub-bundle or a line quotient bundle. This gives our rank three bundle on X. In special situations, the line sub-bundle maps to zero in the line quotient-bundle and this gives a rank two bundle on X. In our attempt to find non-split bundles, we try to ensure that $\mathcal{G}$ itself is non-split. One method for this is to use Frobenius pull-backs in characteristic p. This is why we need finite characteristics at many points of the paper.

Throughout the paper we describe very explicitly how to carry out the constructions using matrices. The explicit nature of the constructions make them easy to implement on a computer algebra system. The examples given use very special sections but at least in our constructions of rank three bundles, one can certainly take more general sections and invoke upper semi-continuity to get other examples.

2. Setup

Let X be a smooth, projective variety. Let Y be a divisor on X corresponding to a section $s \in \Gamma(X, \mathcal{O}_X(Y))$. Suppose we have on Y a sequence of vector bundles

$$0 \to A \to F \to B \to 0 \tag{2.1}$$

and assume that F extends to a vector bundle $\mathcal{F}$ on X. Define $\mathcal{G}$ on X as in the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{F} & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & \mathcal{F}(-Y) & = & \mathcal{F}(-Y) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Clearly $\mathcal{G}$ is a vector bundle on X of rank equal to $\operatorname{rank}(A) + \operatorname{rank}(B)$.

Suppose now we have also two bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on X, L_1, L_2 their restrictions to Y, such that there exists a surjection from L_1 to A and a bundle injection from B to L_2. Suppose the maps given thus: $\phi : L_1 \to F$, $\psi : F \to L_2$ also lift to maps $\Phi : \mathcal{L}_1 \to \mathcal{F}$, $\Psi : \mathcal{F} \to \mathcal{L}_2$. (This can always be arranged if $\mathcal{L}_1$ (resp. $\mathcal{L}_2$) is a sum of very negative (resp. positive) powers of an ample line bundle on X.) We get the commutative diagram

$$\begin{array}{ccccc}
L_1 & \xrightarrow{\phi} & F & \xrightarrow{\psi} & L_2 \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{L}_1 & \xrightarrow{\Phi} & \mathcal{F} & \xrightarrow{\Psi} & \mathcal{L}_2
\end{array}$$

Then the composite $\Psi\Phi$ is divisible by the section s and we can construct on X the mapping between vector bundles

$$\mathcal{F}(-Y) \oplus \mathcal{L}_1 \xrightarrow{\Delta} \mathcal{F} \oplus \mathcal{L}_2(-Y)$$

where $\Delta = \begin{bmatrix} sI & \Phi \\ \Psi & s^{-1}\Psi\Phi \end{bmatrix}$.

With this notation, the following results are straightforward.

Proposition 2.1.

(1) *$\mathcal{G}$ equals the image of Δ.*
(2) *The inclusion of $\mathcal{G}$ in $\mathcal{F} \oplus \mathcal{L}_2(-Y)$ is a bundle inclusion.*
(3) *(The split case) If $s^{-1}\Psi\Phi$ vanishes on Y, then $\mathcal{G} \mid_Y \cong A \oplus B(-Y)$.*
(4) *(The trivial case) If $s^{-1}\Psi\Phi = \lambda I, 0 \neq \lambda \in k$, then $\mathcal{L}_1$ is a summand of $\mathcal{G}$.*

Proposition 2.2. (Subbundles of $\mathcal{G}$) *Suppose $\mathcal{F}$ splits as $\mathcal{N} \oplus \mathcal{N}'$, with the induced splitting $F = N \oplus N'$. Suppose there is a map $\theta : N(-Y) \to A$ which can be lifted to a map $\Theta : \mathcal{N}(-Y) \to \mathcal{L}_1$ in such a way that $\Phi\Theta$ has image in $\mathcal{N}'$. Then there is a induced map $\mathcal{N}(-Y) \to \mathcal{G}$ which is a bundle inclusion iff its restriction $N(-Y) \to \mathcal{G}|_Y$ is a bundle inclusion.*

Proof. Consider $\begin{bmatrix} sI & \Phi\Theta & s^{-1}\Psi\Phi\Theta + \Psi.i \end{bmatrix}^\vee : \mathcal{N}(-Y) \to \mathcal{N} \oplus \mathcal{N}' \oplus \mathcal{L}_2(-Y)$ where i is the inclusion. It factors through Δ, giving $\mathcal{N}(-Y) \to \mathcal{G}$ and on $X - Y$ where $s \neq 0$, it is clearly a bundle inclusion. $\square$

Proposition 2.3. (Quotient bundles of $\mathcal{G}$) *Suppose $\mathcal{F}$ splits as $\mathcal{M} \oplus \mathcal{M}'$ with the induced splitting $F = M \oplus M'$. Suppose there is a map $\gamma : B(-Y) \to M$ which lifts to a map $\Upsilon : \mathcal{L}_2(-Y) \to \mathcal{M}$ in such a way that $\Upsilon\Psi$ vanishes on $\mathcal{M}(-Y)$. Then there is an induced map $\mathcal{G} \to \mathcal{M}$ which is a surjection of bundles iff its restriction $\mathcal{G}|_Y \to M$ is a surjection of bundles.*

Proof. Consider $\begin{bmatrix} sI & \Upsilon\Psi & p\Phi + \Upsilon s^{-1}\Psi\Phi \end{bmatrix} : \mathcal{M}(-Y) \oplus \mathcal{M}'(-Y) \oplus \mathcal{L}_1 \to \mathcal{M}$ (p is the projection.) It factors through Δ giving $\mathcal{G} \to \mathcal{M}$ and on $X - Y$, it is a surjection. □

3. Four generated rank two bundles

3.1. **Four generated rank two bundles in general.** Let Y be a scheme with structure sheaf $\mathcal{O}_Y$. Let $\mathcal{O}_Y(e)$ denote the line bundle associated to a Cartier divisor, e, on Y.

Definition 3.1. A vector bundle, B on Y, is said to be **four generated** if there exists a rank four vector bundle F of the form $F = \oplus_{i=1}^4 \mathcal{O}_Y(e_i)$ and a surjective map $F \twoheadrightarrow B$.

Let B be a four generated rank two vector bundle on a scheme Y. Since B is four generated there is an exact sequence $0 \to A \to F \to B \to 0$ (where A is also a rank two vector bundle). Let the map from F to B be given by the matrix $\begin{bmatrix} s_1 & s_2 & s_3 & s_4 \end{bmatrix}$ where the s_i are global sections of twists of B. Let m denote the first Chern class of B. Since B has rank two, we have $B \cong B^\vee(m)$. This induces a composite map $\psi : F \to F^\vee(m)$ whose image is B where ψ is the skew symmetric matrix

$$\psi = \begin{bmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{21} & 0 & s_{23} & s_{24} \\ s_{31} & s_{32} & 0 & s_{34} \\ s_{41} & s_{42} & s_{43} & 0 \end{bmatrix},$$

where $s_{ij} = s_i \wedge s_j$. There is the standard Plücker relation among the s_{ij}'s: $s_{12}s_{34} - s_{13}s_{24} + s_{14}s_{23} = 0$.

A similar argument can be performed on the dual map $F^\vee \xrightarrow{\begin{bmatrix} t_1 & t_2 & t_3 & t_4 \end{bmatrix}} A^\vee \to 0$. Let n denote the first Chern class of the rank two vector bundle A. Then $m + n$ and $e_1 + e_2 + e_3 + e_4$ are linearly equivalent. As before, we get a map $\phi : F^\vee(n) \to F$ with image A and matrix $\phi = (t_{ji})$. The two maps, ϕ and ψ give a complex which is exact at the middle. It is easy to identify ϕ in terms of ψ. Indeed we claim that we may choose

$$\phi = \begin{bmatrix} 0 & s_{43} & -s_{42} & s_{32} \\ s_{34} & 0 & s_{41} & -s_{31} \\ -s_{24} & s_{14} & 0 & s_{21} \\ s_{23} & -s_{13} & s_{12} & 0 \end{bmatrix}.$$

To see this, let ϕ' be the matrix written above. Observe that ϕ' gives a homogeneous map from $F^\vee(n)$ to F and that the product $\psi\phi'$ equals zero due to the Plücker relation. One can check locally that the image of ϕ' is the kernel of ψ since at a point, two of the generators, say s_1 and s_2 generate B with s_3, s_4 dependent on s_1, s_2. Hence $\phi = \phi'$.

These specific choices for ψ and ϕ will be made for most of the later discussions.

3.2. **Four generated rank two bundles on the pseudo-Grassmannian in $\mathbf{P}^5$.**

Definition 3.2. Suppose a, b, c, d, e, f form a regular sequence of positive degree homogeneous forms on $\mathbf{P}^5$. In addition, suppose that $s = af - be + cd$ is homogeneous. We will call the hypersurface defined by $s = 0$ a **pseudo-Grassmannian** in $\mathbf{P}^5$.

Proposition 3.3. *Let Y be a pseudo-Grassmannian in $\mathbf{P}^5$ defined by the equation $s = af - be + cd$. Let $k_1, k_2, k_3, k_4, k_5, k_6$ denote the degrees of a, b, c, d, e, f. Then Y has standard four generated rank two bundles, A and B, with first Chern classes $c_1(B) = k_1 + k_3 - k_5$ and $c_1(A) = k_2 - k_4 - k_6$.*

Proof. Consider the two matrices

$$\Psi = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix}, \Phi = \begin{bmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{bmatrix}.$$

These matrices give maps $\Phi : \mathcal{L}_1 \to \mathcal{F}$ and $\Psi : \mathcal{F} \to \mathcal{L}_2$, where

$$\mathcal{L}_1 = \mathcal{O}_{\mathbf{P}^5}(k_2 - k_4 - k_6) \oplus \mathcal{O}_{\mathbf{P}^5}(-k_6) \oplus \mathcal{O}_{\mathbf{P}^5}(-k_5) \oplus \mathcal{O}_{\mathbf{P}^5}(-k_4) \tag{3.1}$$

$$\mathcal{F} = \mathcal{O}_{\mathbf{P}^5} \oplus \mathcal{O}_{\mathbf{P}^5}(k_2 - k_4) \oplus \mathcal{O}_{\mathbf{P}^5}(k_3 - k_6) \oplus \mathcal{O}_{\mathbf{P}^5}(k_1 - k_5) \tag{3.2}$$

$$\mathcal{L}_2 = \mathcal{O}_{\mathbf{P}^5}(k_2 - k_4 + k_1) \oplus \mathcal{O}_{\mathbf{P}^5}(k_1) \oplus \mathcal{O}_{\mathbf{P}^5}(k_2) \oplus \mathcal{O}_{\mathbf{P}^5}(k_3). \tag{3.3}$$

Both Φ and Ψ have determinant equal to s^2, hence determine sheaves supported on Y. Let L_1, L_2, F be the restrictions of $\mathcal{L}_1, \mathcal{L}_2, \mathcal{F}$ to Y. Let ϕ and ψ denote the restrictions of Φ and Ψ to Y. Both ϕ and ψ have rank exactly two at each point of Y. Let A be the rank two bundle corresponding to the image of ϕ and let B be the rank two bundle corresponding to the image of ψ. Observing that $\Psi\Phi = sI$ we get $\psi\phi = 0$. Thus we have an exact sequence of bundles on Y

$$L_1 \xrightarrow{\phi} F \xrightarrow{\psi} L_2$$

which leads to the exact sequence

$$0 \to A \to F \to B \to 0.$$

Thus the quotient rank two bundle B on Y is four-generated.

The map $\psi : F \to L_2$ on Y has image equal to B, hence $\wedge^2\psi$ has image equal to $\wedge^2 B$. Modulo the equation $s = 0$, $\wedge^2\psi$ factors as $\mu^\vee\mu$ where $\mu = \begin{bmatrix} a & -b & c & d & -e & f \end{bmatrix} : \wedge^2 F \to \mathcal{O}_Y(k_1 + k_2 - k_4)$. Since μ is surjective, this serves to show that $\wedge^2 B = \mathcal{O}_Y(k_1 + k_2 - k_4)$. By the homogeneity conditions on s, $\mathcal{O}_Y(k_1 + k_2 - k_4) = \mathcal{O}_Y(k_1 + k_3 - k_5)$. Since $c_1(A) + c_1(B) = c_1(F)$, we have $c_1(A) = k_2 - k_4 - k_6$. □

These pseudo-Grassmannians will be used to produce rank four bundles on the ambient $\mathbf{P}^5$. Using our standard setup from Section 2, we consider the rank four bundle $\mathcal{G}$ on $\mathbf{P}^5$ given by $0 \to \mathcal{G} \to \mathcal{F} \to B \to 0$. This bundle $\mathcal{G}$ is not very interesting because it is actually a sum of line bundles on $\mathbf{P}^5$ (the matrix $s^{-1}\Psi\Phi$ is the identity matrix and the map Δ provides an isomorphism of $\mathcal{G}$ with $\mathcal{L}_1$). However, in characteristic p, the Frobenius morphism from Y to Y can be used to get a pullback of B which is still a four-generated rank two bundle on Y.

Proposition 3.4. *Let Y be a pseudo-Grassmannian in $\mathbf{P}^5$.*
1) In characteristic p, let $B^{(r)}$ be the pullback of the rank two quotient bundle by the r-th power of the Frobenius morphism on Y, $r \geq 1$. Let $\mathcal{G}$ be the rank four bundle on $\mathbf{P}^5$ that is obtained from

$$0 \to \mathcal{G} \to \mathcal{F}^{(r)} \to B^{(r)} \to 0.$$

Then $\mathcal{G}$ is not a sum of line bundles.
2) In arbitrary characteristic, suppose that Y is reducible with Y_1 as one of its components. Let $\mathcal{G}$ be the rank four bundle on $\mathbf{P}^5$ that is obtained from

$$0 \to \mathcal{G} \to \mathcal{F} \to B \otimes \mathcal{O}_{Y_1} \to 0.$$

Then $\mathcal{G}$ is not a sum of line bundles.

Proof. For 1), let $q = p^r$. We will denote by $()^{(r)}$ a pull back by the power of the Frobenius morphism. Note that $s^{-1}\Psi^{(r)}\Phi^{(r)} = s^{q-1}I$, so $\mathcal{G}$ is the image of the map

$$\Delta^{(r)} = \begin{bmatrix} sI & \Phi^{(r)} \\ \Psi^{(r)} & s^{q-1}I \end{bmatrix} : \mathcal{F}^{(r)}(-Y) \oplus \mathcal{L}_1^{(r)} \to \mathcal{F}^{(r)} \oplus \mathcal{L}_2^{(r)}(-Y).$$

For 2), let s_1 be the equation of Y_1 and let $s = s_1 s_2$. Then $\mathcal{G}$ is the image of the map

$$\Delta_1 = \begin{bmatrix} s_1 I & \Phi \\ \Psi & s_2 I \end{bmatrix} : \mathcal{F}(-Y_1) \oplus \mathcal{L}_1 \to \mathcal{F} \oplus \mathcal{L}_2(-Y_1).$$

In either case, $\mathcal{G}$ is the image of a minimal matrix, Υ, from $\mathbf{P}$ to $\mathcal{Q}$ (where $\mathbf{P}$ and $\mathcal{Q}$ are both the sum of eight line bundles). For notational convenience, we will write

$$\Upsilon = \begin{bmatrix} s_1 I & \Phi \\ \Psi & s_2 I \end{bmatrix}.$$

Note that $\Phi\Psi = \Psi\Phi = s_1 s_2 I$. Let

$$\Upsilon_1 = \begin{bmatrix} s_2 I & -\Phi \\ \Psi & -s_1 I \end{bmatrix}.$$

Note that $\Upsilon_1 \Upsilon = 0$. Hence the pair Υ, Υ_1 gives a complex which is exact in the middle by reasons of rank. Since $\Upsilon_1^\vee$ can be transformed to Υ by row and column operations, we have an exact sequence

$$0 \to \mathcal{G} \to \mathcal{Q} \to \mathcal{G}^\vee \to 0.$$

If $\mathcal{G}$ is a sum of line bundles, so is $\mathcal{G}^\vee$, hence the map $\mathcal{G} \to \mathcal{Q}$ is a split injection. An identical argument on the other side proves that if $\mathcal{G}$ is a sum of line bundles, then the map $\mathcal{P} \to \mathcal{G}$ is a split surjection. But now the composite map $\Upsilon : \mathcal{P} \to \mathcal{Q}$ cannot be minimal. Hence, $\mathcal{G}$ is not a sum of line bundles. □

Remark 3.5. We remark that if we take $k_6 = 0$ and $f = 0$, and we do the same constructions on $\mathbf{P}^4$, with a, b, c, d, e a regular sequence on $\mathbf{P}^4$, then all of the algebraic facts described above go through unchanged, even though Y is now a 'cone': $be - cd = 0$. In this case also, when we perform pullbacks by Frobenius or restrict to a component of Y, the result of Proposition 3.4 still holds and we get a rank four bundle $\mathcal{G}$ on $\mathbf{P}^4$ which is not a sum of line bundles.

3.3. Four generated rank two bundles on $\mathbf{P}^3$.

Proposition 3.6. Construction of a family of four generated rank two bundles on $\mathbf{P}^3$

Let T, U, V, W be a regular sequence of positive degree homogeneous forms in $\mathbf{P}^3$ with degrees t, u, v, w and with $t + w = u + v$. Let $r \geq 2$ be an integer. Let $a = T^r, b = U^r, e = V^r, f = W^r, c = TW - UV, d = \Sigma_{i=0}^{r-1} (-1)^i (TW)^{r-i-1}(UV)^i$. Denote the degrees of a, b, c, d, e, f by $k_1, k_2, k_3, k_4, k_5, k_6$. Then

1. $af - be + cd = 0, k_1 = rt, k_2 = ru, k_3 = t + w, k_4 = (r-1)(t + w), k_5 = rv, k_6 = rw$.
2. $\Psi\Phi = 0$ *where*

$$\Psi = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix}, \Phi = \begin{bmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{bmatrix}.$$

3. Ψ, Φ *fit into an exact sequence*

$$\mathcal{L}_1 \xrightarrow{\Phi} \mathcal{F} \xrightarrow{\Psi} \mathcal{L}_2$$

where

$$\mathcal{L}_1 = \mathcal{O}_{\mathbf{P}^3}(k_2 - k_4 - k_6) \oplus \mathcal{O}_{\mathbf{P}^3}(-k_6) \oplus \mathcal{O}_{\mathbf{P}^3}(-k_5) \oplus \mathcal{O}_{\mathbf{P}^3}(-k_4) \tag{3.4}$$

$$\mathcal{F} = \mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(k_2 - k_4) \oplus \mathcal{O}_{\mathbf{P}^3}(k_3 - k_6) \oplus \mathcal{O}_{\mathbf{P}^3}(k_1 - k_5) \tag{3.5}$$

$$\mathcal{L}_2 = \mathcal{O}_{\mathbf{P}^3}(k_2 - k_4 + k_1) \oplus \mathcal{O}_{\mathbf{P}^3}(k_1) \oplus \mathcal{O}_{\mathbf{P}^3}(k_2) \oplus \mathcal{O}_{\mathbf{P}^3}(k_3). \tag{3.6}$$

4. $A = \text{im}(\Phi)$ *and* $B = \text{im}(\Psi)$ *are four generated rank two bundles on* $\mathbf{P}^3$.

Proof. The proof of 1), 2), 3) is clear. To prove 4) one needs only to check that all of the entries of $\wedge^3\Psi$ and $\wedge^3\Phi$ are zero and that the ideal of entries of $\wedge^2\Psi$ and $\wedge^2\Phi$ are each $\mathfrak{m}$-primary. □

The purpose of Proposition 3.6 is to demonstrate the abundance of four generated rank two bundles on $\mathbf{P}^3$. The set of four generated bundles described by the proposition is by no means a complete description of all such bundles.

4. Construction of rank three bundles

4.1. Rank three bundles on $\mathbf{P}^4$ in any characteristic.

In this section we will utilize the same notations and constructions as were utilized in section 3. In Proposition 3.6 we saw that there exist many four generated non-split rank two bundles on $\mathbf{P}^3$. From any four generated non-split rank two bundle, we can construct non-split rank three bundles on $\mathbf{P}^4$. Cohomology considerations demonstrate that these bundles are distinct from Tango's rank three bundles and from extensions of the Horrocks-Mumford bundle.

Let B be a (non-split) four generated rank two bundle on $\mathbf{P}^3$. We can associate to B the following: homogeneous forms a, b, c, d, e, f in $\mathbf{P}^3$, a four generated rank two bundle A and a pair of matrices

$$\Psi = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{bmatrix}, \Phi = \begin{bmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{bmatrix}.$$

In addition, we have an exact sequence

$$\mathcal{L}_1 \xrightarrow{\Phi} \mathcal{F} \xrightarrow{\Psi} \mathcal{L}_2$$

where $A = \mathrm{im}(\Phi)$, $B = \mathrm{im}(\Psi)$ and $\mathcal{F} = \mathcal{O}_{\mathbf{P}^3}(e_1) \oplus \mathcal{O}_{\mathbf{P}^3}(e_2) \oplus \mathcal{O}_{\mathbf{P}^3}(e_3) \oplus \mathcal{O}_{\mathbf{P}^3}(e_4)$ for some integers e_1, e_2, e_3, e_4. If $m = c_1(B)$ and $n = c_1(A)$ then $\mathcal{L}_1 = \mathcal{F}^\vee(n)$ and $\mathcal{L}_2 = \mathcal{F}^\vee(m)$.

Let $\mathbf{P}^3$ be defined in $X = \mathbf{P}^4$ by the equation $h = 0$, h a linear form on $\mathbf{P}^4$. Let $Y = \{h^t = 0\}$ be the t-th order standard thickening of $\mathbf{P}^3$ in $\mathbf{P}^4$. Let $\pi : Y \to \mathbf{P}^3$ be projection from a point not on Y, and let us indicate the pullbacks to Y of our various bundles and matrices on $\mathbf{P}^3$ by the same letters. In particular, the matrices ϕ, ψ are the same matrices on $\mathbf{P}^3$ and on Y.

All the free bundles (F, L_1, L_2) and matrices can be lifted to $\mathbf{P}^4$, with a canonical choice of $\Phi = \phi$ and $\Psi = \psi$ on $\mathbf{P}^4$. In this choice of lift, $\Psi\Phi = 0$. Hence on $\mathbf{P}^4$, we obtain a map between vector bundles:

$$\mathcal{F}(-t) \oplus \mathcal{F}^\vee(n) \xrightarrow{\Delta} \mathcal{F} \oplus \mathcal{F}^\vee(m-t).$$

Here $\Delta = \begin{bmatrix} h^t I & \phi \\ \psi & 0 \end{bmatrix}$. The image of this map is a rank four bundle $\mathcal{G}$ on $\mathbf{P}^4$.

<u>Line sub-bundle of $\mathcal{G}$</u>: It is evident that the sum of the first and fifth columns is nowhere vanishing on $\mathbf{P}^4$ (as are the sum of the second and sixth, the sum of the third and seventh, the sum of the fourth and eighth). However we need to be able to add columns in a homogeneous fashion.

For example, in order to add columns one and five, we need $e_1 - t = -e_1 + n$. For this, since our choice of t is open, we need $t = 2e_1 - n > 0$. Now n is the first Chern class of A and since $A \hookrightarrow F = \oplus_{i=1}^4 \mathcal{O}_{\mathbf{P}^3}(e_i)$, it follows that (since A is non-split) that $n < e_i + e_j$ when $i \neq j$. Hence the inequality $e = 2e_1 - n > 0$ is satisfied if e_1 is not the smallest of the e_i's. By this reasoning, we see that $\mathcal{G}$ has at least three different rank three quotients for different choices of pairs of columns and corresponding choice of the positive integer t.

<u>Line quotient bundle of $\mathcal{G}$</u>: For this, we look at the sum of the first and fifth rows of Δ (or second and sixth, etc) which is evidently nowhere vanishing on $\mathbf{P}^4$. We ask when such a row is homogeneous. This occurs when $e_1 = -e_1 + m - t$. Hence we need $t = m - 2e_1 > 0$. Once again, since $F \to B \to 0$ and B is non-split, it is guaranteed that this is true as long as e_1 is not the largest of the four. Thus we can always find a t such that $\mathcal{G}$ will contain a rank three sub-bundle.

4.2. **Rank three bundles on $\mathbf{P}^5$ in characteristic p.** Psuedo-Grassmannians give rise in an easy way to rank three bundles on $\mathbf{P}^5$ in characteristic p. There are not many rank three bundles known on $\mathbf{P}^5$. One example in characteristic zero is the 'parent bundle' of Horrocks and variants of this bundle [6], [2]. Other examples can be built by taking extensions of Tango's rank two bundle on $\mathbf{P}^5$ (in characteristic two) by a line bundle. Cohomology considerations demonstrate that the rank three bundles constructed in this section differ, in general, from any specializations of known bundles to characteristic p.

Start with a pseudo-Grassmannian Y in $X = \mathbf{P}^5$ using the notation of Proposition 3.3. Let p be the characteristic of the field and let $\mathcal{G}$ on $\mathbf{P}^5$ be the (non-trivial) rank four bundle obtained in Proposition 3.4 using pull-backs by the r-th power of Frobenius ($r \geq 1$).

In order to find a nowhere vanishing section of $\mathcal{G}$, one can try to ensure that the first and fifth columns of $\Delta^{(r)}$ can be added in a homogeneous fashion. Indeed, observe that the sum of these two columns is

$$\begin{bmatrix} s & -f^q & e^q & -d^q & s^{q-1} & a^q & b^q & c^q \end{bmatrix}^\vee,$$

which is a regular sequence on $\mathbf{P}^5$, hence it would be nowhere vanishing if it were homogeneous. Furthermore under such circumstances, since $\mathcal{G}$ is nontrivial, the quotient rank three bundle will not be a sum of line bundles.

The requirement for homogeneity is that $-k_1 - k_6 = q(k_2 - k_4 - k_6)$. Hence in order to construct a non-split rank three bundle on $\mathbf{P}^5$ in characteristic p, we may fix $r \geq 1$ and find six forms $a, \dots, f$ with degrees $k_1, \dots, k_6$ subject to the conditions

$$k_1 + k_6 = k_2 + k_5 = k_3 + k_4 = q(k_4 + k_6 - k_2).$$

This can be accomplished for any p and any $r \geq 1$. To do this, fix k_2, k_4, k_6 such that $k_4 + k_6 - k_2 > 0$ and such that $q(k_4 + k_6 - k_2)$ is larger that the largest of k_2, k_4, k_6, and then calculate k_1, k_3, k_5.

Likewise, rank three subbundles of $\mathcal{G}$ will exist upon suitable restrictions on the degrees $k_1, k_2, \dots, k_6$. For example, if we insist that the second and sixth rows of $\Delta^{(r)}$ be added in homogeneous fashion, the numerical requirements are that

$$q(k_2 - k_4) = qk_1 - (k_1 + k_6),$$

and the row is clearly nowhere vanishing on $\mathbf{P}^5$. The kernel of this surjection from $\mathcal{G}$ to $\mathcal{O}_{\mathbf{P}^5}(q(k_2 - k_4))$ is a rank three sub-bundle of $\mathcal{G}$.

4.3. **Rank three bundles on $\mathbf{P}^4$ using pseudo-Grassmannians in characteristic zero.** In characteristic zero where the Frobenius morphism does not exist, we can try to mimic the last construction as follows. Let Y be a pseudo-Grassmannian in $\mathbf{P}^5$ or its hyperplane section in $\mathbf{P}^4$, with its usual rank two bundle B and matrices Φ, Ψ over the ambient projective space $\mathbf{P}^n$. The image of the matrix Δ is a sum of line bundles on $\mathbf{P}^n$. But now suppose that Y has a proper irreducible component Y_1 with the equation s of Y factoring as

$$s = s_1 s_2.$$

Consider the restriction, B_1, of B to Y_1. It is four generated on Y_1 and it therefore gives rise to a rank four bundle $\mathcal{G}_1$ on $\mathbf{P}^n$. By Proposition 3.4, $\mathcal{G}_1$ is not split and it is given as the image of the matrix

$$\Delta_1 = \begin{bmatrix} s_1 I & \Phi \\ \Psi & s_2 I \end{bmatrix} : \mathcal{F}(-Y_1) \oplus \mathcal{L}_1 \to \mathcal{F} \oplus \mathcal{L}_2(-Y_1).$$

We can now explicitly look for line sub and quotient bundles of $\mathcal{G}_1$ using the matrix Δ_1.

The factorization of the equation s of Y is easy to achieve on $\mathbf{P}^4$ (we have no example, on $\mathbf{P}^5$, of a regular sequence of 6 forms for which $af - be + cd$ factors). Here is an example:

Let a, b, c, d, e, f be chosen as $X_0^2, X_1^2, X_2^2, X_3^2, X_4^2, 0$ in our matrix Φ. The equation of Y factors as

$$X_1^2 X_4^2 - X_2^2 X_3^2 = (X_1 X_4 + X_2 X_3)(X_1 X_4 - X_2 X_3).$$

Taking Y_1 from the first factor, our matrix Δ_1 is a matrix of quadratic forms. We can add columns (or rows) without violating homogeneity requirements, and it is evident that the sum of the first and fifth columns (or rows) is nowhere vanishing on $\mathbf{P}^4$. It is fairly easy to see that this particular rank three bundle is isomorphic to the simplest example of the construction in 4.1, when we start out with the null correlation bundle on $\mathbf{P}^3$ (which is four generated).

5. Construction of rank two bundles

In this section we construct some rank two bundles on $\mathbf{P}^5$ in characteristic two, $\mathbf{P}^4$ in finite characteristics, and on the quadric S_5 in $\mathbf{P}^6$ in any characteristic. Our basic observation is that when we obtain a rank four bundle $\mathcal{G}$ on an ambient space, we saw previously that it may have a line sub-bundle $\mathcal{N}$ or a line quotient-bundle $\mathcal{M}$. In this section, we will find situations where the two phenomena occur simultaneously and in fact give a monad

$$0 \to \mathcal{N} \to \mathcal{G} \to \mathcal{M} \to 0,$$

whose homology yields a rank two bundle.

5.1. **Rank two bundles on $\mathbf{P}^4$ in characteristic 2.** The null-correlation bundle on $\mathbf{P}^3$ is four generated and hence gives rise to a rank four bundle $\mathcal{G}$ on $\mathbf{P}^4$. In 4.1, we determined the criterion ($t = 2$) for the sum of the first and fifth columns of the matrix Δ to give a line sub-bundle $\mathcal{N}$ of $\mathcal{G}$. This sum looks like $\begin{bmatrix} 0 & s_{34} & -s_{24} & s_{23} & h^t & s_{21} & s_{31} & s_{41} \end{bmatrix}^\vee$. Likewise, we had found that various pairs of rows can be added to give a line quotient bundle, again with $t = 2$. The composite $\mathcal{L} \to \mathcal{G} \to \mathcal{M}$ can never be zero when the line bundle quotient obtained from the sum of the first and fifth rows (being $0 + h^t$). However, the quotient $\mathcal{M}$ obtained from the sum of the second and sixth rows will yield a monad if it is true that $s_{34} + s_{21} = 0$. This can be arranged in characteristic two (essentially this says that the quadrics $X_0X_1 + X_2X_3$ and $X_0X_1 - X_2X_3$ are the same). Hence this gives a rank two bundle on $\mathbf{P}^4$ in characteristic two.

Though we will shortly give many more examples of rank two bundles on $\mathbf{P}^4$ in finite characteristics, the following geometric argument justifies the claim of the last paragraph. Consider $G_2(4)$ in $\mathbf{P}^5$. It has Plücker equation $X_0X_1 + X_2X_3 + X_4X_5$. Intersect $G_2(4)$ with $X_4 = X_5$. The quadric we get in $\mathbf{P}^4$ has equation $X_0X_1 + X_2X_3 + X_4^2$. In characteristic two, there is a purely inseparable map from $\mathbf{P}^3$ to this quadric ($(x_0, x_1, x_2, x_3) \mapsto (x_0^2, x_1^2, x_2^2, x_3^2, (x_0x_1 + x_2x_3))$). Pull back the universal quotient bundle on $G_2(4)$ to $\mathbf{P}^3$ to get a four generated rank two bundle B (a null-correlation bundle with Chern class $m = 2$). The six s_{ij}'s for B satisfy one linear relation $s_{12} = s_{34}$ since the map to $G_2(4)$ factors through $G_2(4) \cap \mathbf{P}^4$.

5.2. **Rank two bundles on $\mathbf{P}^5$ in characteristic 2.** Suppose that Y is a pseudo-Grassmannian in $\mathbf{P}^5$ in characteristic two. Using the notation of Proposition 3.4, let $\Delta^{(1)}$ be the matrix $\begin{bmatrix} sI & \Phi^{(1)} \\ \Psi^{(1)} & sI \end{bmatrix}$ which is obtained after pull-back by the standard Frobenius. Let $\mathcal{G}$ be the rank four image of this matrix.

We again ask for criteria for the sum of the first and fifth columns of $\Delta^{(1)}$ to give a line sub-bundle $\mathcal{N}$ of $\mathcal{G}$. The numerical conditions were obtained in Section 4.2. If we inspect the sum of these two columns, since the first and fifth entries are s, it is possible to add the first and fifth rows of $\Delta^{(1)}$ and maintain homogeneity. Therefore, let $\mathcal{M}$ be the quotient-bundle of $\mathcal{G}$ so obtained. We have created a monad, and hence a rank two bundle on $\mathbf{P}^5$.

Explicitly, choose integers $l > 0, m \geq 0, n \geq 0$. Let $k_1 = l + 2m + n, k_2 = l, k_3 = l + m + 2n, k_4 = l + m, k_5 = l + 2m + 2n, k_6 = l + n$. One checks that this yields a situation where $\Delta^{(1)}$, through the sum of its first and fifth columns and its first and fifth rows, gives the monad for a rank two bundle on $\mathbf{P}^5$.

Tango's rank two bundle on $\mathbf{P}^5$ [9] is obtained when Y is the standard Grassmannian $G_2(4)$ in $\mathbf{P}^5$ ($l = 1, m = n = 0$). One can check that the bundle obtained on $\mathbf{P}^4$ in Section 5.1 is a restriction of Tango's rank two bundle. On the other hand, when $l = 1, m = 2, n = 3$, we get $k_1, \ldots, k_6$ equal to $8, 1, 9, 3, 11, 4$ and the bundle obtained is not a usual pull-back of Tango's rank two bundle.

5.3. **Rank two bundles on $\mathbf{P}^4$ in characteristic p.** Let $q = p^r$. We saw above that in characteristic two and for the usual Frobenius pullback, we could construct a monad on $\mathbf{P}^5$, using $\mathcal{G}$, a line sub-bundle and a line quotient-bundle built out of the first and fifth rows and the first and fifth columns of $\Delta^{(1)}$. It is fairly clear that these choices of rows and columns will not work in characteristic $p \neq 2$ or even in characteristic two but with higher Frobenius pullbacks. However, we can hope to use a different pair of rows, say the second and sixth.

To see what happens, suppose that we have successfully used the the sum of the first and fifth columns of $\Delta^{(r)}$ to obtain a line sub-bundle $\mathcal{N}$. This imposes a degree requirement. The sum of the columns is

$$\begin{bmatrix} s & -f^q & e^q & -d^q & s^{q-1} & a^q & b^q & c^q \end{bmatrix}^{\vee}.$$

In order for the second and sixth rows to complete a monad, we require the relation $-f^q + a^q = 0$ (with the additional degree requirement $k_1 = k_6$). Such a relation cannot hold on $\mathbf{P}^5$ since the pseudo-Grassmannian Y is defined by a regular sequence a, b, c, d, e, f on $\mathbf{P}^5$.

However, if we choose a, f to be appropriate powers of linear forms, say $a = X_4^{k_1}, f = X_5^{k_6}$, then on the hyperplane $\mathbf{P}^4 : X_4 - X_5 = 0$, the restricted bundle $\mathcal{G}|_{\mathbf{P}^4}$ admits a line sub-bundle and quotient bundle which compose to zero giving a monad on $\mathbf{P}^4$. Thus arranging degrees in this fashion, we get rank two bundles on $\mathbf{P}^4$, in characteristic p.

In order to build such bundles, for any r, with $q = p^r$, find positive integers $k_1, k_2, \ldots, k_6$ satisfying the conditions $k_1 + k_6 = k_2 + k_5 = k_3 + k_4 = q(k_4 + k_6 - k_2)$ of Section 4.2 and also $k_1 = k_6$. This can always be done. One set of choices would be: let $k_1 = qu, (u > 0)$, $k_4 = v$ where v is any positive integer $< (q + 2)u$. Then take $k_2 = v + (q - 2)u, k_3 = 2qu - v, k_5 = (q + 2)u - v, k_6 = k_1$, and we have a solution.

As an example, let us take $\mathbf{P}^5$ in characteristic 5 and let $q = 25$. Choose a, b, c, d, e, f a regular sequence of degrees $25, 24, 49, 1, 26, 25$ where $a = X_4^{25}, f = X_5^{25}$, with $s = af - be + cd$. $s = 0$ defines a hypersurface Y of degree 50. Using pull-backs by the square of the Frobenius map, we get a matrix $\Delta^{(2)}$. Using the sum of columns 1 and 5 and the sum of rows 2 and 6, and restricting to $\mathbf{P}^4 : X_4 = X_5$, we get a rank two bundle on $\mathbf{P}^4$ with a monad

$$0 \to \mathcal{O}_{\mathbf{P}^4}(50) \to \mathcal{G}|_{\mathbf{P}^4} \to \mathcal{O}_{\mathbf{P}^4}(575) \to 0.$$

5.4. **Rank two bundles on S_5 in arbitrary characteristic.** The next bundle we describe is well known in the literature. It is described in [9], as the starting point in his construction of a rank two bundle on $\mathbf{P}^5$ in characteristic two.

Let us denote the standard quadric $G_2(4)$ in $\mathbf{P}^5$ by S_4 and the standard five-dimensional quadric in $\mathbf{P}^6$ by S_5. With linear coordinates a, b, c, d, e, f, g on $\mathbf{P}^6$ and with $\mathbf{P}^5$ given by $g = 0$, S_5 has equation $af - be + cd = g^2$. and S_4 is $g = 0$ on S_5.

Viewing the pair S_5, S_4 as the X, Y of Section 2, we produce a rank four bundle $\mathcal{G}$ on S_5 using the four generated universal rank two bundle B on S_4. The matrices Φ, Ψ on S_5 can be chosen to be the same as the standard matrices ϕ, ψ on S_4. On S_5, the product $\Psi\Phi = g^2 I$. We hence get a matrix Δ on S_5 of the form

$$\Delta = \begin{bmatrix} gI & \Phi \\ \Psi & gI \end{bmatrix}.$$

This matrix Δ gives a map $8\mathcal{O}_{S_5}(-1) \to 8\mathcal{O}_{S_5}$. It is immediate to see that the sum of the first and fifth columns and the difference of the first and fifth rows yields a monad

$$0 \to \mathcal{O}_{S_5}(-1) \to \mathcal{G} \to \mathcal{O}_{S_5} \to 0.$$

The homology of this monad gives a rank two bundle $\mathcal{E}$ on S_5. It remains to see that $\mathcal{E}$ is not the sum of two line bundles. This can be seen as follows: if the map $\mathcal{G} \to \mathcal{O}_{S_5}$ is not a split surjection, then $\mathcal{E}$ has non-zero H^1, hence is not a sum of line bundles (because S_5 has Picard group $\mathbf{Z}$). But if $\mathcal{G} \to \mathcal{O}_{S_5}$ is a split surjection, so is its restriction to S_4: $A \oplus B(-1) \to \mathcal{O}_{S_4}$. This can happen only if one of the maps $A \to \mathcal{O}_{S_4}$ or $B(-1) \to \mathcal{O}_{S_4}$ is split surjective. But this is not possible on the Grassmann variety S_4.

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