ARITHMETICALLY COHEN-MACAULAY BUNDLES
ON THREE DIMENSIONAL HYPERSURFACES

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Abstract. We prove that any rank two arithmetically Cohen-Macaulay vector bundle on a general hypersurface of degree at least six in \( \mathbb{P}^4 \) must be split.

1. Introduction

An arithmetically Cohen-Macaulay (ACM for short) vector bundle on a hypersurface \( X \subset \mathbb{P}^n \) is a bundle \( E \) for which \( H^i(X, E(k)) = 0 \) for \( 0 < i < n - 1 \) and for all integers \( k \). ACM bundles of large rank, which are not split as a sum of line bundles, exist on any hypersurface \( X \) of degree \( > 1 \) (see [2]), and it is also conjectured by Buchweitz-Greuel-Schreyer (op. cit.) that low rank ACM bundles on smooth hypersurfaces should be split. For example, it is well known [7] that there are no non-split ACM bundles of rank two on a smooth hypersurface in \( \mathbb{P}^6 \). In [6], it was proved that on a general hypersurface of degree \( \geq 3 \) in \( \mathbb{P}^5 \), there are no non-split ACM bundles of rank two.

In the current paper, we extend this result to general hypersurfaces of degree \( d \geq 6 \) in \( \mathbb{P}^4 \):

Main Theorem. Fix \( d \geq 6 \). There is a non-empty Zariski open set of hypersurfaces of degree \( d \) in \( \mathbb{P}^4 \), none of which support an indecomposable ACM rank two bundle.

The special case when \( d = 6 \) was proved by Chiantini and Madonna [3]. The result we prove is optimal and we refer the reader to [6] for more details.

Our result can also be translated into a statement about curves on \( X \): on a general hypersurface \( X \) in \( \mathbb{P}^4 \) of degree \( d \geq 6 \), any arithmetically Gorenstein curve on \( X \) is a complete intersection of \( X \) with two other hypersurfaces in \( \mathbb{P}^4 \). Yet another translation of the result is that the defining equation of such a hypersurface cannot be expressed as the Pfaffian of a skew-symmetric matrix in a non-trivial way.
In the current paper, we will need some of the results from [6]. We will also use the relation between rank two ACM bundles on hypersurfaces and Pfaffians that was observed by Beauville in [1] and which was not needed in [6].

As usual, let $H^i_c(X, E)$ denote the graded module $\oplus_{k \in \mathbb{Z}} H^i_c(X, E(k))$. The following theorem summarizes and paraphrases some of the results from [6] that are important for the proof of the Main Theorem above.

**Theorem 1** ([6] Thm 1.1(3), Cor 2.3). Let $E$ be an indecomposable rank two ACM bundle on a smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^4$. Then $H^2_c(X, \mathcal{E}nd(E))$ is a non-zero cyclic module of finite length, with the generator living in degree $-d$. If $d \geq 5$ and $X$ is general, then $H^2_c(X, \mathcal{E}nd(E)) = 0$.

### 2. ACM bundles and Pfaffians

We work over an algebraically closed field of characteristic zero. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree $d$. Let $E$ be an ACM vector bundle of rank two on $X$. By Horrocks’ criterion [5], this is equivalent to saying that $E$ has a resolution,

$$0 \to F_1 \xrightarrow{\Phi} F_0 \xrightarrow{\sigma} E \to 0,$$

where the $F_i$’s are direct sums of line bundles on $\mathbb{P}^n$. We will assume that this resolution is minimal, with $F_0 = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^n}(-a_i)$ where $a_1 \leq a_2 \leq \cdots \leq a_n$. Using [1], we may write $F_1$ as $F_0^\vee(e - d)$, where $e$ is the first Chern class of $E$, and we may assume that $\Phi$ is a skew-symmetric $n \times n$ matrix with $n$ even. The $(i,j)$-th entry $\phi_{ij}$ of $\Phi$ has degree $d - e - a_i - a_j$. The condition of minimality implies that there are no non-zero scalar entries in $\Phi$ and thus every degree zero entry must be zero.

We quote some facts about Pfaffians and refer the reader to [9] for more details. Let $\Phi = (\phi_{ij})$ be an $n \times n$ even-sized skew symmetric matrix and let $\text{Pf}(\Phi)$ denote its Pfaffian. Then $\text{Pf}(\Phi)^2 = \det \Phi$. Let $\Phi(i,j)$ be the matrix obtained from $\Phi$ by removing the $i$-th and $j$-th rows and columns. Let $\Psi$ be the skew-symmetric matrix of the same size with entries $\psi_{ij} = (-1)^{i+j}\text{Pf}(\Phi(i,j))$ for $0 \leq i < j \leq n$. We shall refer to $\text{Pf}(\Phi(i,j))$ as the $(i,j)$-Pfaffian of $\Phi$. The product $\Phi \Psi = \text{Pf}(\Phi)I_n$, where $I_n$ is the identity matrix.

**Example 1.** Let $n = 4$ above. Then

$$\text{Pf}(\Phi) = \phi_{12}\phi_{34} - \phi_{13}\phi_{24} + \phi_{14}\phi_{23}.$$
The following lemma shows the relation between skew-symmetric matrices, ACM rank 2 bundles and the equation defining the hypersurface.

**Lemma 1.** Let $E$ be a rank 2 ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d$ and let $\Phi : F_1 \to F_0$ be the minimal skew-symmetric matrix associated to $E$. Then $X = X_\Phi$, the zero locus of $\text{Pf}(\Phi)$. Conversely, let $\Phi : F_1 \to F_0$ be a minimal skew-symmetric matrix such that the hypersurface $X_\Phi$ defined by $\text{Pf}(\Phi)$ is smooth of degree $d$. Then $E_\Phi$, the cokernel of $\Phi$, is a rank 2 ACM bundle on $X_\Phi$.

**Proof.** Let $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ be the polynomial defining $X$. Since $E$ is supported along $X$, $\det \Phi = f^n$ for some $n$ up to a non-zero constant where $\Phi$ is as in resolution (1). Locally $E$ is a sum of two line bundles and so the matrix $\Phi$ is locally the diagonal matrix $(f, f, \cdots, 1)$. Since the determinant of this diagonal matrix is $f^2$, we get $f = \text{Pf}(\Phi)$ up to a non-zero constant.

To see the converse: let $\Phi$ be any skew-symmetric matrix and $\Psi$ be defined as above. Let $f = \text{Pf}(\Phi)$ be the Pfaffian. Since $\Phi \Psi = f I_n$, this implies that the composite $F_0(-d) \xrightarrow{f} F_0 \to E_\Phi$ is zero. Thus $E_\Phi$ is annihilated by $f$ and so is supported on the hypersurface $X_\Phi$ defined by $f$. Since $X_\Phi$ is smooth, by the Auslander-Buchsbaum formula, $E_\Phi$ is a vector bundle on $X_\Phi$. Therefore locally $\Phi$ is a diagonal matrix of the form $(f, \cdots, f, 1, \cdots, 1)$ where the number of $f$'s in the diagonal is equal to the rank of $E$. Since $\det(\Phi) = f^2$, we conclude that rank of $E_\Phi$ is 2. \qed

Let $V \subset \text{Hom}(F_1, F_0)$ be the subspace consisting of all minimal skew-symmetric homomorphisms, where $F_i$'s are as above. The following is an easy consequence of the above lemma.

**Lemma 2.** Let $\Phi_0 \in V$ be an element such that $E_{\Phi_0}$ is a rank 2 ACM bundle on a smooth hypersurface $X_{\Phi_0}$. Then there exists a Zariski open neighbourhood $U$ of $\Phi_0$ such that for any $\Phi \in U$, $X_\Phi$ is a smooth hypersurface and $E_\Phi$ is a rank two ACM bundle supported on $X_\Phi$.

### 3. Special cases

The proof of the Main Theorem will require the study of some special cases, which are listed below.

**Lemma 3.** Consider the following three types of curves in $\mathbb{P}^4$:

- a curve $C$ which is the complete intersection of three general hypersurfaces, two of which are of degree $\leq 2$. 
• a curve $D$ which is the locus of vanishing of the principal $4 \times 4$ sub-Pfaffians of a general $5 \times 5$ skew-symmetric matrix $\chi$ of linear forms.

• a curve $C_r, r \geq 0$, which is the locus of vanishing of the $2 \times 2$ minors of a general $4 \times 2$ matrix $\Delta$ with one row consisting of forms of degree $1 + r$, and the remaining three rows consisting of linear forms.

The general hypersurface $X$ in $\mathbb{P}^4$ of degree $\geq 6$ cannot contain any curve of the first two types. The general hypersurface $X$ of degree $d \geq \max\{6, r + 4\}$ cannot contain any curve of the third type.

Proof. The curve $C$ is smooth if the hypersurfaces are general. If $\chi$ is general, the curve $D$ is smooth (see [10], page 432 for example). If $\Delta$ is general, the curve $C_r$ is smooth (see op. cit. page 425).

The proof of the lemma is a straightforward dimension count. By counting the dimension of the set of all pairs $(Y, X)$ where $Y$ is a smooth curve of the described type and $X$ is a hypersurface of degree $d$ containing $Y$, it suffices to show that this dimension is less than the dimension of the set of all hypersurfaces $X$ of degree $d$ in $\mathbb{P}^4$. This can be done by showing that if $S$ denotes the (irreducible) subset of the Hilbert scheme of curves in $\mathbb{P}^4$ parameterizing all such smooth curves $Y$, then the dimension of $S$ is at most $h^0(\mathcal{O}_Y(d)) - 1$.

This argument was carried out in [8] where $Y$ is any complete intersection curve in $\mathbb{P}^4$. The case where $Y$ equals the first type of curve $C$ in the list above is Case 2 of [8]. Hence we will only consider the types of curves $D$ and $C_r$ here.

If $Y$ is of type $D$ in the list, the sheaf $\mathcal{I}_D$ has the following free resolution ([10], page 427):

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 5} \xrightarrow{\chi} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 5} \to \mathcal{I}_D \to 0.$$  

Computing Hilbert polynomials, we see that $D$ is a smooth elliptic quintic in $\mathbb{P}^4$, and it easily computed that that $h^0(N_D) = 25$. Since $h^0(\mathcal{O}_D(d)) = 5d, for d \geq 6$, we get $\dim S \leq h^0(N_D) \leq h^0(\mathcal{O}_D(d)) - 1$.

If $Y$ is of type $C_r$ in the list, we may analyze the dimension of the parameter space of all such $C_r$’s as follows. Let $S$ be the cubic scroll in $\mathbb{P}^4$ given by the vanishing of the two by two minors of the linear $3 \times 2$ submatrix $\theta$ of the $4 \times 2$ matrix $\Delta$. The ideal sheaf of the determinantal surface $S$ has resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-3)^2 \xrightarrow{\theta} \mathcal{O}_{\mathbb{P}^n}(-2)^3 \to \mathcal{I}_S \to 0.$$  

From this one computes the dimension of the set of such cubic scrolls to be 18, since the 30 dimensional space of all $3 \times 2$ linear matrices is acted
on by automorphisms of $\mathcal{O}_{\mathbb{P}^n}(-3)^2$ and $\mathcal{O}_{\mathbb{P}^n}(-2)^3$, with scalars giving the stabilizer of the action. Furthermore, by dualizing the resolution, we get a resolution for $\omega_S$:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-5) \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 3} \overset{\theta'}{\to} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2} \to \omega_S \to 0.$$  

A section of $\omega_S(r+3)$ gives a lift $\mathcal{O}_{\mathbb{P}^n}(-r-3) \overset{\alpha}{\to} \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 2}$, and we obtain a $4 \times 2$ matrix $\left( \begin{array}{c} \theta \\ \alpha' \end{array} \right)$ of the required type. Hence $C_r$ is a curve on $S$ in the linear series $|K_S + (r+3)H|$, where $H$ is the hyperplane section on $S$. Intersection theory on $S$ gives $K_S.K_S = 8$, $K_S.H = -5$ and $H.H = 3$. Using this, we may compute the dimension of the linear system of $C_r$ on $S$, and we get the dimension of the set $S$ of all such $C_r$ in $\mathbb{P}^4$ to be $21$ if $r = 0$ and $(3/2)r^2 + (13/2)r + 24$ otherwise.

The ideal sheaf of $C_r$ has a free resolution given by the Eagon-Northcott complex [4]

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-r-4)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^n}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(-r-3)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^n}(-r-2)^{\oplus 3} \to \mathcal{I}_{C_r} \to 0.$$  

Let $d \geq \max\{6, r + 4\}$ be chosen as in the statement of the lemma. Then $d = r + s + 4$ where $s \geq 0$ ($s \geq 2$ when $r = 0$; $s \geq 1$ when $r = 1$).

Using the above resolution, a calculation gives

$$h^0(\mathcal{O}_{C_r}(d)) = \frac{3}{2}r^2 + \frac{29}{2}r + 3rs + 4s + 17.$$  

The required inequality $\dim S < h^0(\mathcal{O}_{C_r}(d))$ is now evident.  

\Box

4. Proof of Main Theorem

In this section, $E$ will be an indecomposable ACM bundle of rank two and first Chern class $e$ on a smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^4$. The minimal resolution (1) gives $\sigma : F_0 \to E \to 0$, and we may describe $\sigma$ as $[s_1, s_2, \ldots, s_n]$ where $s_1, s_2, \ldots, s_n$ is a set of minimal generators of the graded module $H^0(E)$ of global sections of $E$, with degrees $a_1 \leq a_2 \leq \cdots \leq a_n$.

Lemma 4. If $E$ is an indecomposable rank 2 ACM bundle with first Chern class $e$ on a general hypersurface $X \subset \mathbb{P}^4$ of degree $d \geq 6$, then there is a relation in degree $3 - e$ among the minimal generators of $S^2E$.

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E}nd(E) \to (S^2E)(-e) \to 0.$$
$S^2E(-e)$ has the same intermediate cohomology as $\text{End}(E)$ since the sequence splits in characteristic zero.

Choose a minimal resolution of $S^2E$:

$$0 \to B \to C \to S^2E \to 0,$$

where $C$ is a direct sum of line bundles on $X$ and $B$ is a bundle on $X$ with $H^1(X, B) = 0$.

We first show that $B^\vee(e + d - 5)$ is not regular. For this, consider the dual sequence $0 \to (S^2E)^\vee \to C^\vee \to B^\vee \to 0$.

By Serre duality and Theorem 1

$$H^1(X, (S^2E)^\vee(d + e - 5)) = 0.$$

Therefore

$$H^0(X, C^\vee(d + e - 5)) \to H^0(X, B^\vee(d + e - 5))$$

is onto. If $B^\vee(d + e - 5)$ were regular, the same would be true for

$$H^0(X, C^\vee(d + e - 5 + k)) \to H^0(X, B^\vee(d + e - 5 + k)) \quad \forall k \geq 0.$$

However, this is false for $k = d$ since by Serre duality and Theorem 1,

$$H^1(X, (S^2E)^\vee(2d + e - 5)) \neq 0.$$  

Thus $B^\vee(e + d - 5)$ is not regular. Now

$$H^1(X, B^\vee(e + d - 6)) \cong H^2(X, B(1 - e)) \cong H^1(X, S^2E(1 - e))$$

$$\cong H^1(X, \text{End}(E)(1)).$$

By Serre duality,

$$H^1(X, \text{End}(E)(1)) \cong H^2(X, \text{End}(E)(d - 6))$$

which by Theorem 1 equals zero for $d \geq 6$ (this is the main place where we use the hypothesis that $d \geq 6$). Furthermore, $H^2(X, B^\vee(e + d - 7)) = 0$ since $H^2(X, B) = 0$. Since $B^\vee(e + d - 5)$ is not regular, we must have $H^3(X, B^\vee(e + d - 8)) \neq 0$.

In conclusion, $H^0(X, B(3 - e)) \neq 0$. In other words, there is a relation in degree $3 - e$ among the minimal generators of $S^2E$.

**Lemma 5.** Let $E$ be as above. Then $1 \leq a_1 + a_2 + e \leq a_1 + a_3 + e \leq 2$.

**Proof.** The resolution (1) for $E$ gives an exact sequence of vector bundles on $X$: $0 \to G \to F_0 \to E \to 0$, where $F_0 = F_0 \otimes \mathcal{O}_X$ and $G$ is the kernel. This yields a long exact sequence,

$$0 \to \wedge^2G \to F_0 \otimes G \to S^2F_0 \to S^2E \to 0.$$

From the arguments after Lemma 2.1 of [6] (using formula (5)), it follows that $H^2(\wedge^2G) = 0$. Hence the map $S^2F_0 \to S^2E$ is surjective on global sections. The image of this map picks out the sections $s_i s_j$ of degree $a_i + a_j$ in $S^2E$. Observe that the lowest degree minimal sections
s_1, s_2 of E induce an inclusion of sheaves \(O_X(-a_1) \oplus O_X(-a_2) \xrightarrow{(s_1, s_2)} E\) whose cokernel is supported on a surface in the linear system \(|O_X(a_1 + a_2 + e)|\) on \(X\) (a nonempty surface when \(E\) is indecomposable). Hence \(1 \leq a_1 + a_2 + e\). There is an induced inclusion

\[S^2[O_X(-a_1) \oplus O_X(-a_2)] \hookrightarrow S^2E.\]

Therefore the three sections of \(S^2E\) given by \(s_1^2, s_1s_2, s_2^2\) cannot have any relations amongst them. Since these are also three sections of \(S^2E\) of the lowest degrees, they can be taken as part of a minimal system of generators for \(S^2E\). It follows that the relation in degree 3\(-e\) among the minimal generators of \(S^2E\) obtained in the previous lemma must include minimal generators other than \(s_1^2, s_1s_2, s_2^2\). Since the other minimal generators have degree at least \(a_1 + a_2\), and since we are considering a relation amongst minimal generators, we get the inequality \(a_1 + a_2 \leq 2 - e\). □

**Lemma 6.** For any choice of \(1 \leq i < j \leq n\), the \((i,j)\)-Pfaffian of \(\Phi\), is non-zero. Consequently, its degree (which is \(a_i + a_j + e\)) is at least \((n-2)/2\).

**Proof.** On \(X\), \(E\) has an infinite resolution

\[
\cdots \to F^\vee_0(e - 2d) \xrightarrow{\Phi} F_0(-d) \xrightarrow{\Psi} F^\vee_0(e - d) \xrightarrow{\Phi^\vee} F_0 \to E 
\]

We also have

\[
F^\vee_0(e - 2d) \xrightarrow{\Phi} F_0(-d) \xrightarrow{\sigma} E(-d) \xrightarrow{\alpha} F^\vee_0(e - d) \xrightarrow{\Phi^\vee} F_0.
\]

Let \(\overline{\Theta} = \sigma^\vee \alpha \sigma\). Since \(\sigma = (s_1, \ldots, s_n)\), we may express the \((i,j)\)-th entry of \(\overline{\Theta}\) as \(\theta_{ij} = s_i^\vee s_j\) (suppressing the canonical isomorphism \(\alpha\)). \(\Phi^\vee = -\Phi\) and \(\alpha \sigma : F_0(-d) \to E^\vee(e - d)\) is surjective on global sections. Hence we have a commuting diagram

\[
\begin{array}{cccccc}
F^\vee_0(e - 2d) & \xrightarrow{\Phi} & F_0(-d) & \xrightarrow{\sigma} & E(-d) & \xrightarrow{\alpha} \\
\downarrow \cong & & & & & \\
F^\vee_0(e - d) & \xrightarrow{\Phi^\vee} & F_0 & \to & E & \to 0
\end{array}
\]

It is easy to see that \(B\) is an isomorphism. As a result, every column of \(B\) has a non-zero scalar entry.

Now suppose that \(\overline{\psi}_{ij} = 0\) for some \(i, j\) so that \(\sum_k s_i^\vee s_k b_{kj} = 0\). Let \(Y_i\) be the curve given by the vanishing of the minimal section \(s_i\) with
the exact sequence
\[ 0 \to \mathcal{O}_X(-a_i) \xrightarrow{s_i} E \xrightarrow{s'_i} I_{Y_i/X}(a_i + e) \to 0. \]
Hence \( s'_i s_i = 0 \) and \( s'_i s_k \) for \( k \neq i \) give minimal generators for \( I_{Y_i/X} \).

It follows that no \( b_{kj} \) can be a non-zero scalar for \( k \neq i \). Hence \( b_{ij} \) has to be a non-zero scalar and the only one in the \( j \)-th column. However, \( \overline{\psi}_{ij} = 0 \). So by the same argument, \( b_{jj} \) is the only non-zero scalar. To avoid contradiction, \( \overline{\psi}_{ij} \) and hence \( \psi_{ij} \neq 0 \) for \( i \neq j \).

We now complete the proof of the Main Theorem. As in the previous lemmas, assume that \( X \) is general of degree \( d \geq 6 \), with \( E \) an indecomposable rank two ACM bundle on \( X \). We will show that the inequalities of Lemma 5 lead us to the special cases of Lemma 3, giving a contradiction.

Let \( \mu = a_1 + a_2 + e \). By Lemma 5, \( 1 \leq \mu \leq 2 \).

**Case** \( \mu = 1 \). In this case, in order for the \((1, 2)\)-Pfaffian of \( \Phi \) to be linear, by Lemma 6, \( n \) must equal 4. In the \( 4 \times 4 \) matrix \( \Phi \), the \((1, 2)\)-Pfaffian is the entry \( \phi_{34} \) which we are claiming is linear. Likewise the \((1, 3)\)-Pfaffian is the entry \( \phi_{24} \) which by Lemma 5 has degree \( a_1 + a_3 + e \leq 2 \). By Lemma 2, we may assume that \( \phi_{14}, \phi_{24}, \phi_{34} \) define a smooth complete intersection curve and \( X \) contains this curve by example 1.

By Lemma 3, \( X \) cannot be general.

**Case** \( \mu = 2 \). In this case \( a_2 = a_3 \). By Lemma 6, \( n \) must be 4 or 6. The case \( n = 4 \) is ruled out again by the arguments of the above paragraph since \( \Phi \) has two entries of degree 2 in its last column. We will therefore assume that \( n = 6 \). The matrix
\[
\Phi = \begin{pmatrix}
0 & \phi_{12} & \phi_{13} & \phi_{14} & \phi_{15} & \phi_{16} \\
* & 0 & \phi_{23} & \phi_{24} & \phi_{25} & \phi_{26} \\
* & * & 0 & \phi_{34} & \phi_{35} & \phi_{36} \\
* & * & * & 0 & \phi_{45} & \phi_{46} \\
* & * & * & * & 0 & \phi_{56} \\
* & * & * & * & * & 0
\end{pmatrix}
\]
is skew-symmetric and by our choice of ordering of the \( a_i \)'s, the degrees of the upper triangular entries are non-increasing as we move to the right or down.

As remarked before, the degree of \( \phi_{ij} \) is \( d - e - a_i - a_j \). The \((1, 2)\)-Pfaffian (which is a non-zero quadric when \( \mu = 2 \)) is given by the expression (see example 1)
\[
Pf(\Phi(1, 2)) = \phi_{34}\phi_{56} - \phi_{35}\phi_{46} + \phi_{36}\phi_{45}.
\]
We shall consider the following two sub-cases, one where $\phi_{56}$ has positive degree (and hence can be chosen non-zero by Lemma 2) and the other where it has non-positive degree (and hence is forced to be zero):

$$d - e - a_5 - a_6 > 0.$$ 

Since $\phi_{34} \cdot \phi_{56}$ is one term in the $(1, 2)$-Pfaffian of $\Phi$, and since degree $\phi_{34}$ is at least degree $\phi_{56}$, they are both forced to be linear. Therefore $\phi_{34}, \phi_{35}, \phi_{36}$ have the same degree (=1) and so $a_3 = a_4 = a_5$. Likewise, $a_4 = a_5 = a_6$. Hence $\Phi$ has a principal $5 \times 5$ submatrix $\chi$ (obtained by deleting the first row and column in $\Phi$) which is a skew symmetric matrix of linear terms, while its first row and first column have entries of degree $1 + r$, $r \geq 0$.

By Lemma 2 we may assume that the ideal of the $4 \times 4$ Pfaffians of $\chi$ defines a smooth curve $C$. $X$ is then a degree $d = 3 + r$ hypersurface containing $C$. By Lemma 3, $X$ cannot be general when $d \geq 6$.

$$d - e - a_5 - a_6 \leq 0.$$ 

In this case, the entry $\phi_{56} = 0$. Suppose $\phi_{46}$ is also zero. Then both $\phi_{36}$ and $\phi_{45}$ must be linear and non-zero since the $(1, 2)$-Pfaffian of $\Phi$ (see equation 2) is a non-zero quadric. Since $a_2 = a_3$, $\phi_{26}$ is also linear. Thus using Lemma 2, $X$ contains the complete intersection curve given by the vanishing of $\phi_{16}$ and the two linear forms $\phi_{36}, \phi_{26}$. By Lemma 3, $X$ cannot be general.

So we may assume that $\phi_{46} \neq 0$. Since $\phi_{35}$ is also non-zero, both must be linear. Hence $a_3 + a_5 = a_4 + a_6$, and so $a_3 = a_4$ and $a_5 = a_6$.

After twisting $E$ by a line bundle, we may assume that $a_2 = a_3 = a_4 = 0 \leq a_5 = a_6 = b$. The linearity of the entry $\phi_{46}$ gives $d - e - b = 1$. The condition $d - e - a_5 - a_6 \leq 0$ yields $1 \leq b$. Taking first Chern classes in resolution (1) gives $e = 2 - a_1$.

Let $r = -a_1$, $s = b - 1$. Then $r, s \geq 0$, and $d = r + s + 4$. If we inspect the matrix $\Phi$, the non-zero rows in columns 5 and 6 give a $4 \times 2$ matrix $\Delta$ with top row of degree $1 + r$ and the other entries all linear. By Lemma 2, we may assume that the $2 \times 2$ minors of this $4 \times 2$ matrix define a smooth curve $C$, as described in Lemma 3. Since $X$ contains this curve, $X$ cannot be general when $d \geq 6$.

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