Buchsbaum bundles on $\mathbf{P}^n$.

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In this note, we investigate whether there exist indecomposable rank two bundles $\mathcal{E}$ on $\mathbf{P}^n$ for which the first cohomology module $H^1_*(\mathcal{E})$ is 1- or 2-Buchsbaum. By this terminology, we mean that the module is annihilated by the irrelevant ideal or the square of the irrelevant ideal. For the purposes of this note, we will also call such a bundle itself a '1-Buchsbaum' or '2-Buchsbaum' bundle. It is well known that the null-correlation bundle is the only such 1-Buchsbaum bundle on $\mathbf{P}^3$. Here, we also establish that mathematical instantons with second Chern class equal to two are the only other 2-Buchsbaum bundles on $\mathbf{P}^3$. The question is also addressed on higher projective spaces. Similar questions have been asked by others. For example, Chang ([C]) shows that there are no smooth Buchsbaum varieties of codimension two in $\mathbf{P}^6$ or higher. The condition of being a Buchsbaum variety is stronger than the condition we have taken up. For the bundle $\mathcal{E}$ that the subvariety corresponds to, it means that all the intermediate cohomology modules of $\mathcal{E}$ as well as the intermediate cohomology of all possible restrictions of $\mathcal{E}$ to linear subspaces are 1-Buchsbaum, in our notation. Recently, Ellia and Sarti ([E-S]) have extended Chang's question to show that there are no smooth 2-Buchsbaum varieties of codimension two in $\mathbf{P}^6$ or higher, where this again is a strong condition imposed on all cohomology modules of all restrictions.

The advantage of these stronger conditions is that they reduce the problem to a problem on $\mathbf{P}^3$ where the question can be studied more easily. Our observation here is that vanishing theorems allow us to concentrate just on the H^1_*-module and control it for restrictions to various linear subspaces. Our conclusion is that 1-Buchsbaum rank two bundles do not exist on $\mathbf{P}^n$ for $n \geq 4$, and 2-Buchsbaum rank two bundles do not exist on $\mathbf{P}^n$ for $n \geq 5$. The question of 2-Buchsbaum bundles on $\mathbf{P}^4$ is open. The questions for $\mathbf{P}^3$ are answered, as we mentioned above, and the question for $\mathbf{P}^2$ is rather trivial and is left unsaid.

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Let $\mathcal{E}$ be a rank two vector bundle on $\mathbf{P}^n_k$, with k an algebraically closed field of characteristic zero, $n \geq 3$. We will suppose that $\mathcal{E}$ is regular. For any hyperplane $H = 0$ in $\mathbf{P}^n$, the restricted bundle $\mathcal{E}_H$ is also regular.

Consider $H^2(\mathcal{E}(-5+a))$. Let Y be the smooth zero-scheme of a general section of $\mathcal{E}$. If c is the first Chern class of $\mathcal{E}$, we have the resolution

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{E} \to \mathcal{O}_{\mathbf{P}^n}(c) \to \mathcal{O}_Y(c) \to 0.$$

So $\omega_Y = \mathcal{O}_Y(c-n-1)$ and

$$H^2(\mathcal{E}(-5+a)) \subseteq H^1(\omega_Y(n+1-5+a)) = H^{n-3}(\mathcal{O}_Y(5-n-1-a))^\vee.$$

By Kodaira vanishing,

1.1) On $\mathbf{P}^5$ and higher, $H^2(\mathcal{E}(-5+a)) = 0$ for $a \geq 0$.

1.2) On $\mathbf{P}^4$, $H^2(\mathcal{E}(-5+a)) = 0$ for $a \geq 1$.

1.3) On $\mathbf{P}^3$, $H^2(\mathcal{E}(-5+a)) = 0$ for $a \geq 2$.

1.4) If $\mathcal{E}$ on $\mathbf{P}^4$ is the restriction of a bundle $\mathcal{F}$ on $\mathbf{P}^5$, where $\mathcal{F}$ is also regular, so that $Y = X \cap \mathbf{P}^4$, then for the smooth threefold X, $H^1(\mathcal{O}_X) = 0$ by the Barth-Lefschetz theorem, hence also $H^1(\mathcal{O}_Y) = 0$, hence

$H^2(\mathcal{E}(-5+a)) = 0$ for $a \geq 0$.

1.5) If $\mathcal{E}$ on $\mathbf{P}^3$ is the restriction of a bundle $\mathcal{F}$ on $\mathbf{P}^4$, where $\mathcal{F}$ is also regular, then $Y = X \cap \mathbf{P}^3$, where the surface X is connected. This translates to $H^3(\mathcal{F}(-5)) = 0$, and hence from the restriction exact sequence,

$H^2(\mathcal{E}(-5+a)) = 0$ for $a \geq 1$.

1.6) If $\mathcal{E}$ on $\mathbf{P}^3$ is the restriction of a regular bundle on $\mathbf{P}^5$, let $\mathcal{F}$ be the intermediate restriction to $\mathbf{P}^4$. Then the surface X is linearly normal, while $\mathcal{F}$ itself satisfies (1.4), hence

$H^2(\mathcal{E}(-5+a)) = 0$ for $a \geq 0$.

1.7)<u>Theorem</u>: Let $\mathcal{E}$ on $\mathbf{P}^4$ or higher be a regular rank two bundle. Then the graded module $M = H^1_*(\mathcal{E})$ has no non-zero summand in gradings -2 or higher. If $\mathcal{E}$ is on $\mathbf{P}^5$ or higher, M has no non-zero summand in gradings -3 or higher.

<u>Proof</u>: We consider $\mathcal{E}$ on $\mathbf{P}^4$ or higher, a regular rank two bundle. $M = H^1_*(\mathcal{E})$ is a graded module over the polynomial ring S. Suppose it has a non-zero summand N in gradings -2 and higher. Restricting $\mathcal{E}$ to a hyperplane $H = 0$, the S/H module $H^1_*(\mathcal{E}_H)$ has a non-zero submodule $N/H.N$ which is a summand of the submodule $M/H.M$. By (1.2), $H^1_*(\mathcal{E}_H)$ equals $M/H.M$ in degrees -3 and higher. Hence $H^1_*(\mathcal{E}_H)$ has $N/H.N$ as a non-zero summand in degrees -2 and higher. Using (1.5), we may restrict all the way down

to a plane $\mathbf{P}^2$ to obtain a restriction $\mathcal{E}_0$ of $\mathcal{E}$ which has the property that the appropriate quotient module $\bar{N}$ is a non-zero summand of $H^1_*(\mathcal{E}_0)$.

Likewise consider $\mathcal{E}$ on $\mathbf{P}^5$ or higher, a regular rank two bundle. Suppose M has a non-zero summand N in gradings -3 and higher. Using (1.1), (1.4) and (1.6), we will obtain a restricted bundle $\mathcal{E}_0$ on $\mathbf{P}^2$ such that $H^1_*(\mathcal{E}_0)$ has a non-zero summand given by the appropriate quotient $\bar{N}$.

Now, a rank two bundle $\mathcal{E}_0$ on $\mathbf{P}^2$ is determined by its cohomology module as follows ([R]): Let $L_1 \to L_0 \to H^1_*(\mathcal{E}_0) \to 0$ be a minimal presentation. Then if G is the kernel of this presentation, $\mathcal{E}_0$ is G sheafified. But any such kernel has rank at least two and so if $H^1_*(\mathcal{E}_0)$ has two summands, the rank of $\mathcal{E}_0$ is violated. In conclusion, $H^1_*(\mathcal{E}_0)$ must be $\bar{N}$.

This creates a problem on $\mathbf{P}^3$. If $\mathcal{E}_1$ is this bundle, $H^2_*(\mathcal{E}_1)$, if non-zero, must make a non-zero contribution to $H^1_*(\mathcal{E}_0)$, from the restriction exact sequence. Hence $H^2_*(\mathcal{E}_1)$ must be zero, whence also $H^1_*(\mathcal{E}_1)$, and so $\mathcal{E}_1$ is split by Horrocks' theorem. This leads to a contradiction, since now $\mathcal{E}$ is also split. ♣

1.8) <u>Corollary</u>: Let $\mathcal{E}$ be an indecomposable rank two bundle on $\mathbf{P}^4$ or higher. Then $H^1_*(\mathcal{E})$ cannot be 1-Buchsbaum.

<u>Proof</u>: Let us twist $\mathcal{E}$ until $\mathcal{E}$ is regular, but $\mathcal{E}(-1)$ is not. A result of Ein ([E]) tells us that $H^1(\mathcal{E}(-2) \neq 0$. If the module is annihilated by the irrelevant ideal, then $H^1(\mathcal{E}(-2)$ is a non-zero module direct summand in degree -2. This violates the theorem above. ♣

1.9) <u>Corollary</u>: Let $\mathcal{E}$ be an indecomposable rank two bundle on $\mathbf{P}^5$ or higher. Then $H^1_*(\mathcal{E})$ cannot be 2-Buchsbaum.

<u>Proof</u>: As above, twist $\mathcal{E}$ until $\mathcal{E}$ is regular, but $\mathcal{E}(-1)$ is not so that $H^1(\mathcal{E}(-2) \neq 0$. Consider V_1, the image of the multiplication map $H^0(\mathcal{O}_{\mathbf{P}^n}(1)) \otimes H^1(\mathcal{E}(-4)) \to H^1(\mathcal{E}(-3))$, and let V_2 be a vector space complement to V_1. Then, if $H^1_*(\mathcal{E})$ is 2-Buchsbaum, $V_2 \oplus H^1(\mathcal{E}(-2)$ is a non-zero module direct summand in degrees -3 and higher. This violates the theorem above. ♣

1.10) <u>Proposition</u>: The only indecomposable rank two bundles on $\mathbf{P}^3$ for which $H^1_*(\mathcal{E})$ is 2-Buchsbaum are the null-correlation bundles (really 1-Buchsbaum) and the mathematical instantons with second Chern class two.

<u>Proof</u>:To start, let us remark that rank two bundles on $\mathbf{P}^3$ which have a 1-Buchsbaum $H^1_*(\mathcal{E})$ module are well understood: up to twists by line bundles, the only such bundle is the null-correlation bundle. See for example [C], Proposition 4.1. We now suppose that $\mathcal{E}$ is a rank two bundle on $\mathbf{P}^3$ which has a 2-Buchsbaum $H^1_*(\mathcal{E})$ module. Observing the proof of (1.9) above, we see that $M = H^1_*(\mathcal{E})$ can be decomposed into a direct sum of indecomposable 2-Buchsbaum (or 1-Buchsbaum) modules and that any summand will

have all its generators in the same degree. Let N be any summand with say a generators in degree 0 (for convenience), s generating relations in degree 1 and t additional generating relations in degree 2. Since N is 2-Buchsbaum, it is zero in degree two, hence $10a \leq 4s+t$. Thus

$$*) \qquad s+t \geq 2a + \frac{1}{2}a + \frac{3}{4}t.$$

Hence when $a \geq 5$, $s+t > 2a+2$, for $a = 3, 4$, $s+t \geq 2a+2$ (with strict inequality if $t \neq 0$), for $a = 2, s+t \geq 2a+1$, and for obvious reasons, when $a = 1, s+t \geq 2a+2$.

The structure theory for minimal monads says that $\mathcal{E}$ is the cohomology of the sheafified complex $L_0^\vee(c) \to L_1 \xrightarrow{\alpha} L_0$, where α gives a minimal presentation of M and where $L_1^\vee \cong L_1(-c)$, c being the first Chern class of $\mathcal{E}$ ([R]). In particular, the rank of L_1 is two plus twice the rank of L_0. Combining this with the bounds on $s+t$, we see that if M has more than one non-zero summand, it can have only two summands both with 2 generators and just 5 linear relations. We claim this cannot happen. For if $L_0 = 2S(p) \oplus 2S(q)$, then $L_1 = 5S(p-1) \oplus 5S(q-1)$. The duality on L_1 leads to $p+q = c+2$, hence $L_0^\vee(c) = 2S(q-2) \oplus 2S(p-2)$. For such a complex, α cannot be the presentation of a 2-Buchsbaum module. In fact, if $p = q$, then we calculate M_{-q+2} and find it has dimension ≥ 4, while if $p > q$, the summand N obtained from $2S(q)$ will have N_{-q+2} of dimension ≥ 2.

Thus M is non-split with a generators in the same degree and $2a+2$ minimal relations in the next two degrees. By our earlier estimates, $a \leq 4$. When $a = 3, 4$, we must have $s = 2a+2, t = 0$, hence we get the monad of a mathematical instanton: $aS(p-2) \to (2a+2)S(p-1) \to aS(p)$. A quick calculation shows us that now for $a = 2, 3$, $M_{p+2} \neq 0$, hence M is not 2-Buchsbaum.

The case $a = 2$ (and $s+t = 6$) allows us two possibilities with $t = 0$ and $t \neq 0$. The first yields the monad $2S(p-2) \to 6S(p-1) \to 2S(p)$ which gives, after a twist, a mathematical instanton with $c_1 = 0, c_2 = 2$. These exist and are all 2-Buchsbaum [H]. The case $t \neq 0$ cannot occur: for if $s = 0$, then we must have $t \geq 20$, which is too large for the structure theorem, while if both $s, t \neq 0$, by the duality on L_1, $s = t$, hence both equal three, which contradicts (*)

Lastly, if $a = 1$, and $s+t = 4$, we have the cases: (i) $s = 0, t = 4$ which is not 2-Buchsbaum; (ii) $s, t \neq 0$, whence by duality on L_1, $s = t = 2$ which is again not 2-Buchsbaum; and finally (iii) $s = 4, t = 0$ which give us the null-correlation bundle. ♣

1.12) <u>Question</u>: What, if any, are the rank two bundles on $\mathbf{P}^4$ which have a 2-Buchsbaum H^1_* module?

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