Construction of rank two vector bundles on Projective spaces

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1 Introduction

The aim of this article is to construct rank two vector bundles on $\mathbf{P}^4$ and $\mathbf{P}^5$ in positive characteristics. Most of these constructions have been done in earlier papers of the author [MK1], [MK2]. So, the aim of this article is to construct those bundles more explicitly, since few mathematicians expressed a desire to see more transparent constructions than in the above mentioned papers. I thank C. Peterson and A.P. Rao for all the motivations they provided and for their constant encouragement and help.

2 The constructions

We begin with two easy lemmas.

Lemma 2.1 *Let $f, g, h, h' \in R$ where R is any ring. If $h \in (f, g)$ then $hh' \in (f, gh')$. Conversely, if $hh' \in (f, gh')$ and f, h' form a regular sequence in R, then $h \in (f, g)$.*

The proofs are obvious.

Lemma 2.2 *Let $R = k[x, y]/I$ where k is a field and $I = (x^a, y^b, x^m y^n)$ with $0 \leq m < a, 0 \leq n < b$. Then $\dim_k R = mb + na - mn$.*

Proof We may assume without loss of generality that $m \leq n$. Then the proof is by induction on m. If $m = 0$, then $I = (x^a, y^n)$ and so $\dim_k R = na$ and so we are done. Now assume that $m > 0$. Then we have a surjection $R \to k[x, y]/J$ where $J = (x^a, y^b, xy)$. The kernel is generated by xy and one easily checks that the kernel is isomorphic to $k[x, y]/J'$ where $J' = (x^{a-1}, y^{b-1}, x^{m-1}y^{n-1})$. Noting that we can apply induction to the latter and the dimension of $k[x, y]/J$ is just $a + b - 1$, an easy computation will finish the proof.

□

Now we assume that the base field is k of charactersitic $p > 0$. We plan to write down a three generated, height two, locally Cohen-Macaulay but arithmetically non-Cohen-Macaulay subscheme in $\mathbf{P}^4$. This will, by taking the first syzygy, give an indecomposable rank two vector bundle on $\mathbf{P}^4$.

Let x,y,z,t,u be the homogeneous coordinates of $\mathbf{P}^4$. Let $\alpha = yu - tz$, $A = y^p\alpha + t^p x^2$ and $F = z^p y^{p^2} + x^2 A^{p-1}$. Notice that all the above are homogeneous. ($\deg \alpha = 2, \deg A = p+2$ and $\deg F = p(p+1)$.) Now we consider the subscheme X defined by $I = (x^{2p+2}, y^{p(p+1)}, F)$. Clearly it is of height two, supported only along the plane defined by $x = y = 0$. The method we follow is the following, to show that X has the required Cohen-Macaulay properties. We project X to $\mathbf{P}^2$, by the coordinate functions z,t,u. This is a finite map and we show that the scheme theoretic fiber of this map has constant length equal to $2p^2(p+1)$. This will show that X is locally Cohen-Macaulay. Further we show that the fiber over the vertex has length $2p(p+1)^2 > 2p^2(p+1)$ and thus X will not be arithmetically Cohen-Maculay.

Fibers over points with $z = 1$

If $z = 1$, then I claim that X is defined by x^{2p+2} and F. So, let J be the ideal defined by these two equations. Then we must show that $y^{p(p+1)} \in J$. Since $y^{p(p+1)} \equiv -y^p x^2 A^{p-1}$ mod F, we need to show the latter is in J. By lemma 2.1, suffices to show that $y^p A^{p-1} \in (x^{2p}, F)$. Since A, x are relatively prime, by lemma 2.1 again, suffices to show that $y^p A^p \in (x^{2p}, AF)$. But $AF = Ay^{p^2} + x^2 A^p$ and $A^p = y^{p^2}\alpha^p + t^{p^2} x^{2p}$. Thus $(x^{2p}, AF) = (x^{2p}, y^{p^2}(A + x^2\alpha^p))$. So, suffices to show that $y^{p(p+1)}\alpha^p$ belongs to this ideal. Since x, y form a sequence, by lemma 2.1, suffices to show that $y^p\alpha^p \in (x^{2p}, A + x^2\alpha^p)$.

$$A + x^2\alpha^p = y^p\alpha + t^p x^2 + x^2(y^p u^p - t^p) = y^p(\alpha + x^2 u^p).$$

By lemma 2.1, since x, y form as sequence, suffices to show that $\alpha^p \in (x^{2p}, \alpha + x^2 u^p)$. Since,

$$\alpha^p = (\alpha + x^2 u^p)^p - x^{2p} u^{p^2} \in (x^{2p}, \alpha + x^2 u^p),$$

we are done.

Now the map $X \to \mathbf{P}^2$ on the open set $z = 1$ is finite and then X is a complete intersection, we see that the map is flat on this open set. So, to compute the length of any fiber, we need only compute it at some point. So, we will do it over $t = u = 0$. Then $\alpha = 0 = A$ and so the fiber is defined by the equations, x^{2p+2} and $F = y^{p^2}$ and thus the length is as we declared.

Fibers over points with $z = 0$ and $t = 1$

Now, $\alpha = yu, A = y^{p+1}u + x^2$ and $F = x^2 A^{p-1}$. I claim that $x^{2p+2} \in (y^{p(p+1)}, F)$. By applying lemma [2.1], one needs to show that $x^{2p} \in (y^{p(p+1)}, A^{p-1})$ and again this is equivalent by lemma [2.1] to showing that $x^{2p}A \in (y^{p(p+1)}, A^p)$. But $A^p = y^{p(p+1)} + x^{2p}$ and so the above ideal is just $(y^{p(p+1)}, x^{2p})$ and thus clearly $x^{2p}A$ is in this ideal. As, before we need to compute the length of the fibers and so it suffices to compute over any point. So, we compute this over the point $u = 0$. Then $\alpha = 0, A = x^2$ and $F = x^{2p}$. So the ideal is $(y^{p(p+1)}, x^{2p})$ and it is clear that it has the declared length.

Fiber over the point $z = t = 0$ and $u = 1$

Now, $\alpha = y, A = y^{p+1}$ and $F = x^2 y^{p^2-1}$. By lemma 2.2 we see that the colength of the ideal in question is,

$$(2p+2)(p^2-1) + 2p(p+1) - 2(p^2-1) = 2p^2(p+1)$$

as declared.

We are now left with the last case, when all $z = t = u = 0$. Then the ideal in question is $(x^{2p+2}, y^{p(p+1)})$ and its colength is $2p(p+1)^2$ again as declared.

The same argument as above gives that the following scheme (where letters denote variables) will give rise to a rank two vector bundle on $\mathbf{P}^5$. Take I defined by,

$$(x^6, y^6, z^2x^4 + ztx^2y^2 + t^2y^4 + x^2y^2(ux + vy))$$

and one checks that this indeed defines a codimension two subscheme of $\mathbf{P}^5$ whose first syzygy is a rank two indecomposable vector bundle on $\mathbf{P}^5$ in characteristic 2.

3 Deformations of vector bundles on $\mathbf{P}^n$

Now, let $p \geq 3$. Then consider an ideal I as before. Let $\alpha = y^{p-2}u^2 - t^{p-1}z$, $A = y^{p(p-2)}\alpha + t^{p(p-1)}x^2$ and $F = z^p y^{p^2(p-2)} + x^2 A^{p-1}$. These are got by the previous equations, by substituting appropriate powers for the variables y, z, t, u. Now, consider $I = (x^{2p+2}, y^{p(p+1)(p-2)}, F)$. This defines a family of subschemes in $\mathbf{P}^3$, where x is the parameter. Exactly as before, one checks that the first syzygy is a vector bundle of rank two and general member is direct sum of line bundles and the special member is indecomposable.

Finally for the charactersitic 2 example in $\mathbf{P}^4$. Take the ideal I defined by,

$$(x^6, y^6, z^6x^4 + z^3tx^2y^2 + t^2y^4 + x^2y^2(u^4x + v^3y))$$

Notice that, the ideal is homogeneous in the variables, y, z, t, u, v with $\deg x = 0$. Thus, using x as a parameter, we get the required deformation. That is, a family of rank two vector bundles on $\mathbf{P}^4$ in charactersitc 2, such that the general member is a direct sum of line bundles and the special member is indecomposable.

References

[MK1] N. Mohan Kumar, *Smooth Degeneration of Complete Intersection Curves in Positive Characteristic*, Inventiones Mathematicae, **(104)** 1991, 313–319.

[MK2] N. Mohan Kumar, *Construction of rank two vector bundles on* $\mathbf{P}^4$ *in positive characteristics*, Inventiones Mathematicae, **(130)** 1997, 277–286.