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Some Remarks on Prill's Problem

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Abstract.

If $f : X \to Y$ is a non-constant map of smooth curves over $\mathbb{C}$ and if there is a degree two map $\pi : X \to C$ where C is a smooth curve with genus less than that of Y, we show that for a general point $P \in Y$, $f^{-1}(P)$ does not move except possibly in one particular case. In particular, this implies that Prill's problem has an affirmative answer if X as above is hyperelliptic or if f is Galois.

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§1. Introduction

Let $f : X \to Y$ be a finite morphism of non-singular irreducible projective curves over $\mathbb{C}$ of degree d. Let G (resp. g) denote the genus of X (resp. Y). Further assume that $g \geq 2$. Then Prill's problem states that for a general point $P \in Y$, $f^{-1}(P)$ does not move. That is, $\mathrm{H}^0(X, f^*\mathcal{O}_Y(P)) = \mathbb{C}$ (See Arbarello et. al. pp268 [ACGH85]). Since Prill's problem has an affirmative solution if f is cyclic (that is, f is Galois with Galois group cyclic), which will be shown in Proposition 2 and is well known, we will assume that $d \geq 3$, noting that any degree two map is cyclic. One of the consequences stated in [ACGH85] is that if f is as above and Galois and if Prill's problem is false for this f then $h^0(X, f^*\mathcal{O}_Y(P)) > 2$. We will write down a proof of this for completeness in Proposition 3.

Recently, it has been shown by Biswas and Butler in [BB05] that Prill's problem has an affirmative answer if X is hyperelliptic. Our theorem below is a generalization of theirs. Our methods are somewhat different from theirs and might be of independent interest.

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Theorem 1. *Let $f : X \to Y$ be as above. Assume that one has a degree two morphism $\pi : X \to C$ where C is a non-singular curve with genus $\rho < g$. Then either Prill's problem has an affirmative answer for f or f is etale, $g = 2$ and $\rho = 1$.*

In particular, Prills' problem has an affirmative answer if either X is hyperelliptic or if f is Galois.

§2. Preliminaries

Here we collect some results on Prill's problem, which are mostly well known. We fix our notation $f : X \to Y$ to be a finite map of degree d with G (resp. g) denoting the genus of X (resp. Y). Also $g \geq 2$.

Proposition 2. *Let f be cyclic. Then Prill's problem has an affirmative answer.*

Proof. If f is cyclic, then $f_*\mathcal{O}_X$ is a direct sum of line bundles on Y, the eigenspaces for the cyclic group. Thus $f_*\mathcal{O}_X = \oplus_{i=1}^{d} L_i$ and clearly we may assume that $L_1 = \mathcal{O}_Y$, $\deg L_i \leq 0$ and $\mathrm{H}^0(Y, L_i) = 0$ for $i > 1$. Thus it suffices to prove that if $L_2, \ldots, L_d$ are a finite set of line bundles on Y with $\deg L_i \leq 0$ with no sections, then for a general point $P \in Y$, $\mathrm{H}^0(Y, L_i(P)) = 0$ for $i \geq 2$. Thus it suffices to prove that for a single such line bundle $L = L_i$, the set S of points P with $\mathrm{H}^0(Y, L(P)) \neq 0$ is a finite set.

It is clear that if $\deg L < -1$, then S is empty. So, we may assume that $\deg L = 0$ or -1. If it is -1 and if $P \in S$, we see that $L = \mathcal{O}_Y(-P)$. Then for any point $Q \neq P$, $L(Q)$ has no section since no two distinct points can be rationally equivalent. If $\deg L = 0$ and $P \in S$, then $L = \mathcal{O}_Y(Q - P)$ for some point Q. Since $\mathrm{H}^0(L) = 0$, $Q \neq P$. If $P \neq R \in S$, we see that there exists a point R' such that $Q + R \sim P + R'$. This implies that Y is hyperelliptic and if σ is the hyperelliptic involution, S consists of at most two points, $P, \sigma(Q)$. Q.E.D.

The following is essentially the content of the exercise in [ACGH85].

Proposition 3. *Let f as above be Galois and assume that Prill's problem has a negative answer for f. Then for a general point $P \in Y$, $h^0(X, f^*\mathcal{O}_Y(P)) > 2$.*

Proof. On the contrary, assume that $W_P = \mathrm{H}^0(X, f^*\mathcal{O}_Y(P)) = 2$ for a general point $P \in Y$. Let G be the Galois group. Thus G acts on W_P and if for a general point P, the group homomorphism $G \to \mathrm{Aut}\,\mathbb{P}(W_P)$ is not injective, then by continuity, there is a normal subgroup $H \subset G$ which acts trivially on W_P for all P. If we consider the map $f' : X/H \to Y$, we immediately see that for a general point $P \in Y$, $h^0(X/H, f'^*(\mathcal{O}_Y(P))) =$

2. Since f' is Galois with Galois group G/H, we may replace X by X/H and thus assume to start with that the map $G \to \operatorname{Aut} \mathbb{P}(W_P)$ is injective for general $P \in Y$ and the map f itself. But, the section corresponding to $f^{-1}(P)$ is fixed by G and thus $G \subset \operatorname{Aut} \mathbb{A}^1$. We have an exact sequence of groups,

$$1 \to \mathbb{C} \to \operatorname{Aut} \mathbb{A}^1 \to \mathbb{C}^* \to 1.$$

Since G is finite, this implies that G is a subgroup of $\mathbb{C}^*$ and hence cyclic. Now by Proposition 2 we are done. Q.E.D.

The following has been proved in [BB05]. Our proof is somewhat different.

Proposition 4. *If Prill's problem is false for f, then f_*K_X is not generically globally generated. That is, the subsheaf of f_*K_X generated by $\mathrm{H}^0(Y, f_*K_X)$ has rank less than d.*

Proof. Suffices to show that for a general point $P \in Y$ the natural map $\mathrm{H}^0(f_*K_X) \to \mathrm{H}^0(f_*K_{X_{|P}})$ is not onto. Since the latter is a vector space of dimension d and the former is a vector space of dimension G, suffices to show that the kernel $\mathrm{H}^0(f_*K_X(-P))$ has dimension greater than $G-d$. By Serre duality, this is just the dimension of $\mathrm{H}^1(X, f^*(P))$. By Riemann-Roch we have, $h^1(X, f^*(P)) = h^0(f^*(P)) - 1 + G - d$ and by hypothesis $h^0(f^*(P)) > 1$. Q.E.D.

Proposition 5. *Let $f : X \to Y$ be a finite map of non-singular curves. Assume that we have finite morphisms $\phi : Y \to \mathbb{P}^1$ and $\psi : Z \to \mathbb{P}^1$ where Z is a non-singular curve. Further assume that $Z' = Z \times_{\mathbb{P}^1} Y$ is irreducible and we have a morphism $\eta : Z' \to X$ such that the composite $Z' \to X \to Y$ is the natural projection $Z' \to Y$. Then Prill's problem has an affirmative answer for f.*

Proof. If Prill's problem is false for f, clearly it is false for $f' = f \circ \eta$, though Z' may be singular. If $p : Z' \to Z$ and $q : Z' \to Y$ denote the two projections, for any point $P \in Y$, we have, $p_*f'^*(\mathcal{O}_Y(P)) = \psi^*\phi_*(\mathcal{O}_Y(P))$. If we write $\phi_*(\mathcal{O}_Y(P))$ as a direct sum of line bundles $\oplus L_i$, $\mathrm{H}^0(Y, \mathcal{O}_Y(P)) = \mathbb{C}$ implies that one of the $L_i = \mathcal{O}_{\mathbb{P}^1}$ and the others have negative degree. Buth then $\psi^*\phi_*(\mathcal{O}_Y(P))$ is a direct sum of one copy of $\mathcal{O}_Z$ and the rest of negative degree. Thus $\mathrm{H}^0(Z', f'^*(\mathcal{O}_Y(P))) = \mathbb{C}$. Q.E.D.

§3. Proof of Theorem 1

Proof. Write $\pi_*\mathcal{O}_X = \mathcal{O}_C \oplus L$ where L is a line bundle of degree $-m$ on C with $m > 0$. For a point $P \in Y$ we have $V_P = \pi_*f^*(\mathcal{O}_Y(P))$

a rank two vector bundle on C. Also we have the natural inclusion $\pi_*\mathcal{O}_X \to V_P$ using the natural inclusion of $\mathcal{O}_X \subset f^*(\mathcal{O}_Y(P))$. Since the cokernel of this map is a sky-scraper sheaf of length d, we see that $\deg V_P = -m + d$. We have $G = h^1(\pi_*\mathcal{O}_X) = \rho + h^1(L)$. By Riemann-Roch $h^1(L) = m+\rho-1$ and thus $G = 2\rho+m-1$. By Riemann-Hurwitz, we have

$$2\rho + m - 2 = G - 1 \geq d(g-1) = (d-2)(g-1) + 2g - 2$$

and thus, $m \geq (d-2)(g-1) + 2(g-\rho)$. Since $g > \rho$ and $g \geq 2$, this implies $m \geq d$.

We will separate the cases when $m = d$ and $m > d$. We see from the above that if $m = d$, then $g = 2$ and $\rho = 1$, since we have assumed that $d \geq 3$. Also the above inequality from Riemann-Hurwitz must be an equality. That is f is etale.

So, now on we will assume that $m > d$. Then $\deg V_P < 0$ for any $P \in Y$. Let M be the saturation of $\mathcal{O}_C$ in V_P. We have then an exact sequence $0 \to M \to V_P \to M' \to 0$ with M, M' line bundles on C and since $\deg M \geq 0$, $\deg M' < 0$. In particular the map $\mathrm{H}^0(V_P) \to \mathrm{H}^0(M') = 0$ is zero. So $\mathrm{H}^0(M) = \mathrm{H}^0(V_P)$ and if Prill's problem is false for f we have $h^0(V_P) > 1$ for a general $P \in Y$. This implies that the inclusion of $\mathcal{O}_C$ in M is strict.

Consider the map $X \times Y \xrightarrow{(\pi,Id)=\phi} C \times Y$. Let $\Gamma \subset X \times Y$ be the graph of f. Also let $p : C \times Y \to C$ and $q : C \times Y \to Y$ be the natural projections. We have an exact sequence

$$0 \to \mathcal{O}_{X\times Y} \to \mathcal{O}_{X\times Y}(\Gamma) \to \Gamma_{|\Gamma} \to 0.$$

Let D be the image of Γ in $C \times Y$. Then we claim that the map $\Gamma \to D$ is birational. If not, since the composite $\Gamma \to D \xrightarrow{p} C$ is just π which has degree two, we see that $D \to C$ must be birational. But C is smooth and thus $D \to C$ must be an isomorphism. But, we have a morphism $D \xrightarrow{q} Y$ and thus we get a non-constant morphism from $C \to Y$. This is absurd since $\rho < g$. Taking direct images, we get an exact sequence,

$$0 \to \phi_*\mathcal{O}_{X\times Y} \to \phi_*\mathcal{O}_{X\times Y}(\Gamma) \to \phi_*\Gamma_{|\Gamma} \to 0.$$

Notice that $\phi_*\mathcal{O}_{X\times Y}(\Gamma) = E$ is a rank two vector bundle on $C \times Y$ since ϕ is a two to one map. Also $\phi_*\mathcal{O}_{X\times Y}$ is just the pull back of $\mathcal{O}_C \oplus L$ by p. Identifying the pull back of $\mathcal{O}_C$ as $\mathcal{O}_{C\times Y}$ let us look at the inclusion of this sheaf in E and let F be the cokernel. I claim that F has torsion. If it has no torsion, then it is a line bundle outside a finite set of points and thus restricting to a general point $P \in Y$, we get an exact sequence $0 \to \mathcal{O}_C \to E_{|P} = V_P \to F_{|P} \to 0$. Since $F_{|P}$ is assumed to be a line bundle, we see that $\mathcal{O}_C$ is saturated in V_P, which we have seen is not the case. Thus we see that F has torsion. Taking the inverse image of the torsion subsheaf of F in E, we get an exact sequence, $0 \to A \to E \to E/A \to 0$ where $\mathcal{O}_{C\times Y} \subset A$ and this inclusion is strict and E/A is torsion free. It is clear that the composite $p^*L \to E \to E/A$ is an injection. Thus we get an inclusion $A \oplus p^*L \subset E$ and let B be its cokernel. We have a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{C\times Y} \oplus p^*L \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & \phi_*\Gamma_{|\Gamma} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \oplus p^*L \arrow[r] & E \arrow[r] & B \arrow[r] & 0
\end{tikzcd}$$

Thus by snake lemma we get an exact sequence,

$$0 \to A' \to \phi_*\Gamma_{|\Gamma} \to B \to 0$$

where A' is the cokernel of $\mathcal{O}_{C\times Y} \subset A$. Since this inclusion is strict, we see that $A' \neq 0$. Since $\phi_*\Gamma_{|\Gamma}$ is torsion free as an $\mathcal{O}_D$- module we see that A' is supported on all of D. Since $\phi : \Gamma \to D$ is birational we see that $\phi_*\Gamma_{|\Gamma}$ is a line bundle on D for general points of D and thus A' is a line bundle on D at general points of D, since D is a smooth curve generically and these two sheaves are equal at general points of D. This implies that B is supported on a finite set of points of D. But, B is the quotient of a rank two vector bundle by another rank two vector bundle on a smooth surface and thus for homological reasons, either support of B is a divisor or empty. This implies that $B = 0$. So, we get $A \oplus p^*L = E$. Restricting to a general point $P \in Y$ and calling the restriced bundle A_P we see that $A_P \oplus L = V_P$. This implies that $\deg A_P = d$. But, then $\pi^*A_P \subset f^*\mathcal{O}_Y(P)$ and the first line bundle has degree $2d$ and the latter d. This is clearly impossible.

If f is Galois, then the only case left to prove is when $g = 2, \rho = 1$ and f is etale. Then as we saw, $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus L$ with $\deg L = -d$. Thus $\deg V_P = 0$ and $h^0(V_P) > 1$ implies $V_P = \mathcal{O}_Y \oplus \mathcal{O}_Y$. Thus $h^0(V_P) = 2$ for a general point $P \in Y$. This contradicts Proposition 3. Q.E.D.

Corollary 6. *If $f : X \to Y$ has degree 3, then Prill's problem has an affirmative answer.*

Proof. From Theorem 1, we may assume that X is not hyperelliptic. Also note that by Riemann-Hurwitz, since $g \geq 2$, $G \geq 4$. The morphism f induces a morphism $Y \to J^3X$ where J^3X is the variety parametrizing line bundles of degree 3. Also, the image is contained in $W^1_3(X)$ if Prill's problem had a negative answer for this f. Since X is not hyperelliptic, by Martens Theorem [Mar67] (also see pp 191-2 [ACGH85]) we have $\dim W_3^1(X) \leq 0$. Thus image of Y in J^3X is constant. In other words, for any two points $P, Q \in Y$, $f^*(P) \sim f^*(Q)$. This is impossible since $g > 0$. Q.E.D.

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