ON THE GEOMETRY OF GENERALISED QUADRICS

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1. Introduction

We work over the field of complex numbers \( \mathbb{C} \).

Let \( \{ f_0, \cdots, f_n; g_0, \cdots, g_n \} \) be a regular sequence in \( \mathbb{P}^{2n+1} \) and suppose that \( Q = \sum_{i=0}^{n} f_i g_i \) is a homogeneous polynomial. We shall refer to the hypersurface \( X \) defined by \( Q \) as a \textit{generalised quadric}. In this note, we prove that generalised quadrics in \( \mathbb{P}^{2n+1} \) for \( n \geq 1 \) are reduced.

In characteristic \( p > 0 \), it is easy to construct generalised quadrics which are non-reduced. By exploiting this fact, low rank vector bundles were constructed on \( \mathbb{P}^4 \) and \( \mathbb{P}^5 \) in [4]. Furthermore, in characteristic 0, reducible generalised quadrics exist in \( \mathbb{P}^3 \); for instance, the hypersurface defined by \( X^2Y^2 - Z^2U^2 = 0 \), where \( X, Y, Z, U \) are the coordinates of \( \mathbb{P}^3 \), is such a generalised quadric. We do not know any examples of reducible generalised quadrics in higher dimensional projective spaces. However, the question of non-reducedness is settled by our main theorem.

2. Atiyah Class and Chern classes of vector bundles over schemes

Let \( X \) be any scheme and \( E \) be any vector bundle on \( X \). We recall that the \textit{Atiyah class} \( \text{at}(E) \) (see[1]) of the vector bundle \( E \) is the natural extension class

\[
0 \to \Omega^1_X \otimes E \to \mathcal{P}(E) \to E \to 0
\]

where \( \mathcal{P}(E) \) is the \textit{principal parts bundle} of \( E \). Thus \( \text{at}(E) \) is an element of the cohomology group \( H^1(X, \Omega^1_X \otimes \text{End}(E)) \). Starting with this class, one can define \textit{Chern-Hodge classes} \( c_i(E) \in H^i(X, \Omega^i_X) \) as follows (see [3] or for a simpler exposition see [5]).

Consider the composition

\[
(\Omega^1_X \otimes \text{End}(E))^\otimes m \to (\Omega^1_X)^\otimes m \otimes \text{End}(E^\otimes m) \to \Omega^m_X \otimes \text{End}(\wedge^m E) \to \Omega^m_X
\]

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where the last map is induced by the trace map $\text{End}(\wedge E) \to \mathcal{O}_X$. We then define (upto a constant non-zero factor) the Chern-Hodge classes from the composite map below:

$$H^1(X, \Omega^1_X \otimes \text{End}(E)) \otimes \text{at}(E)^m \mapsto H^m(X, (\Omega^1_X \otimes \text{End}(E))^\otimes) \mapsto H^m(X, \Omega^m_X)$$

By convention, $c_0(E) = 1 \in H^0(X, \mathcal{O}_X)$. Furthermore, we let $c(E) = \sum c_i(E)$ which is an invertible element in the graded commutative ring $\bigoplus_i H^i(X, \Omega^i_X)$.

Now let $X$ be any scheme and let $\mathcal{F}$ be a coherent sheaf on $X$ which has a finite resolution by vector bundles $0 \to P^\bullet_X \to \mathcal{F} \to 0$

**Definition 1.** $c(\mathcal{F}) = c(P^\bullet_X) := \prod_k c(P^k_X)^{(-1)^k} \in \bigoplus H^i(X, \Omega^i_X)$.

We recall some basic properties of the Chern-Hodge classes. Let $\mathcal{P}(X)$ be the set of all sheaves on $X$ which have a finite resolution by vector bundles.

**Properties:**

1. For any sheaf $\mathcal{F} \in \mathcal{P}(X)$, $c(\mathcal{F})$ is independent of the resolution.
2. For any short exact sequence of sheaves in $\mathcal{P}(X)$

   $$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

   $$c(\mathcal{F}) = c(\mathcal{F}') c(\mathcal{F}'').$$

3. For any morphism $f : Y \to X$, there is a natural ring homomorphism $f^* : \bigoplus_i H^i(X, \Omega^i_X) \to \bigoplus_i H^i(Y, \Omega^i_Y)$ under which if $E$ is a bundle on $X$, then $f^* c(E) = c(f^* E)$.

4. For any bundle $E$ and a line bundle $\mathcal{L}$, we have

   $$c_r(E \otimes \mathcal{L}) = \sum_{i=0}^r c_i(E) c_1(\mathcal{L}^{r-i})$$

5. If $\mathcal{F} \in \mathcal{P}(X)$ and

   $$0 \to P^\bullet_X \to \mathcal{F} \to 0$$

   is a finite resolution by vector bundles and if $f : Y \to X$ is any morphism, we can define $c_Y(\mathcal{F}) \in H^*(Y, \Omega^\bullet_Y)$ as $c(f^* P^\bullet_X)$. In general, this is not $c(f^* \mathcal{F})$, since this sheaf may not have a finite resolution by vector bundles on $Y$. These coincide if

   $$0 \to f^* P^\bullet_X \to f^* \mathcal{F} \to 0$$

remains exact and thus in this case $c_Y(\mathcal{F}) = c(f^* \mathcal{F})$. 
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(6) For any short exact sequence of sheaves in $\mathcal{P}(X)$

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

on $X$ and a morphism $f : Y \to X$, $c^Y(\mathcal{F}) = c^Y(\mathcal{F}')c^Y(\mathcal{F}'')$.

The following lemma, which is the key lemma, is essentially due to Gruson et.al [2].

**Lemma 1.** Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Consider the restriction maps

$$H^i(\Omega^i_{\mathbb{P}^n}) \overset{\alpha}{\to} H^i(\Omega^i_{X_{\text{red}}})$$

and

$$H^i(\Omega^i_X) \overset{\beta}{\to} H^i(\Omega^i_{X_{\text{red}}}).$$

Then $\text{Im} \beta = \text{Im} \alpha$ for $1 \leq i < n - 1$.

**Proof.** Since $\alpha$ factors through $H^i(\Omega^i_X)$, we only need to show that $\text{Im} \beta \subset \text{Im} \alpha$. Since $X$ is irreducible, we may assume that $X$ is defined by a homogeneous polynomial $f^m, m > 1$ with $f$ irreducible and so $X_{\text{red}}$ is given by the vanishing of $f$. We consider the exact sequence

$$\mathcal{O}_X(-\deg(f^m)) \xrightarrow{d(f^m)} \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

Restricting it to $X_{\text{red}}$, we get

$$\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\text{red}}} \cong \Omega^1_X \otimes \mathcal{O}_{X_{\text{red}}}$$

This implies similar isomorphisms,

$$\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\text{red}}} \cong \Omega^i_X \otimes \mathcal{O}_{X_{\text{red}}},$$

for all $i$.

Since $\alpha$ factors through $H^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\text{red}}})$ and similarly $\beta$ factors through $H^i(\Omega^i_X \otimes \mathcal{O}_{X_{\text{red}}})$, it suffices to prove that the map

$$H^i(\Omega^i_{\mathbb{P}^n}) \overset{\delta}{\to} H^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\text{red}}})$$

is onto by the isomorphism (1) above. We have an exact sequence,

$$0 \to \Omega^i_{\mathbb{P}^n}(-d) \to \Omega^i_{\mathbb{P}^n} \to \Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\text{red}}} \to 0,$$

where $d = \deg f$. Taking cohomologies and noting that $H^j(\Omega^i_{\mathbb{P}^n}(-d)) = 0$ for $j = i, i + 1$, since $1 \leq i < n - 1$, we see that $\delta$ is an isomorphism. \qed

**Lemma 2.** Let $M \subset \mathbb{P}^n$ be a closed subscheme of dimension $r$. Then the natural map,

$$\gamma : H^i(\Omega^i_{\mathbb{P}^n}) \to H^i(\Omega^i_M)$$

is injective for $0 \leq i \leq r$. 
Proof. If \( h \in H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \) is the class of the hyperplane section, then \( H^i(\mathbb{P}^n, \Omega^i_{\mathbb{P}^n}) \) is a one dimensional vector space generated by \( h^i \). Thus, it suffices to show that its image in \( H^i(M, \Omega^i_M) \) is non-zero. If it is zero for some \( i < r \), then \( h^r = h^i h^{r-i} = 0 \in H^r(M, \Omega^r_M) \). A proof of the well known fact that \( h^r \neq 0 \) is sketched in the Appendix. \( \square \)

Lemma 3. Let \( X \subset \mathbb{P}^n \) be an irreducible hypersurface which is not reduced. Let \( \mathcal{F} \) be a coherent sheaf on \( X \) with a resolution \( 0 \to P_X^\bullet \to \mathcal{F} \to 0 \) by vector bundles on \( X \) such that \( 0 \to P_X^\bullet \otimes \mathcal{O}_M \to 0 \) is exact where \( M \subset X_{\text{red}} \) and \( \dim M = r \). Then \( 0 = c_{X_{\text{red}}}^i(\mathcal{F}) \in H^i(X, \Omega^i_{X_{\text{red}}}) \) for \( 1 \leq i \leq \min\{r, n-2\} \).

Proof. Since \( 0 \to P_X^\bullet \otimes \mathcal{O}_M \to 0 \) is exact, \( c^M(\mathcal{F}) = 1 \) by Property 5. From Lemma 1 above, it follows that \( \forall 1 \leq i \leq \min\{r, n-2\} \), there exist classes \( t_i \in H^i(\Omega^i_{\mathbb{P}^n}) \) such that \( \beta(c_i(\mathcal{F})) = \alpha(t_i) \).

Let \( \theta : H^i(\Omega^i_{X_{\text{red}}}) \to H^i(\Omega^i_M) \) be the natural map. Then \( \theta \beta(c_i(\mathcal{F})) = c_i^M(\mathcal{F}) = 0 \) for \( i > 0 \). Thus, \( \theta \alpha(t_i) = 0 \). But, \( \theta \alpha = \gamma \) and by Lemma 2, we get that \( t_i = 0 \) for \( 1 \leq i \leq \min\{r, n-2\} \) and thus

\[
\begin{align*}
    c_{X_{\text{red}}}^i(\mathcal{F}) &= \beta(c_i(\mathcal{F})) = 0 \\
    &\text{for } 1 \leq i \leq \min\{r, n-2\}. \quad \square
\end{align*}
\]

3. Generalised Quadrics

In this section, we apply the results of the previous section to show that generalised quadrics in \( \mathbb{P}^{2n+1} \) for \( n \geq 1 \) are reduced.

Let \( Q \subset \mathbb{P}^{2n+1} \) denote the generalised quadric given by the equation \( \sum_{i=0}^n f_i g_i = 0 \). Let

\[
\begin{align*}
    Z &= Q \cap (f_1 = \cdots = f_n = 0) \\
    L_1 &= Q \cap (f_0 = \cdots = f_n = 0) \\
    L_2 &= Q \cap (g_0 = f_1 = \cdots = f_n = 0)
\end{align*}
\]

Then \( Z = L_1 \cup L_2 \) and we have an exact sequence

\[
0 \to \mathcal{O}_{L_2}(-\deg f_0) \to \mathcal{O}_Z \to \mathcal{O}_{L_1} \to 0
\]

Furthermore, \( Z \) is a complete intersection of \( n \) ample divisors on \( Q \), \( L_i \) for \( i = 1, 2 \) are local complete intersection subschemes in \( Q \) of codimension (and dimension) \( n \).

Theorem 1. The generalised quadric \( Q \) is reduced.
Proof. If $Q$ is not reduced, let $X$ be an irreducible component of $Q$ which is not reduced and let $X_{\text{red}}$ denote the subscheme $X$ with the reduced structure. Thus $\sum f_i g_i = f f'$ with $f$ an irreducible polynomial, $r > 1$ where $f^r = 0$ defines $X$ and $f = 0$ defines $X_{\text{red}}$.

Let $Z' = Z \cap X$, $L'_i = L_i \cap X$. It is easy to see that $Z'$ is a complete intersection in $X$ by $f_i, i > 0$. We consider the Koszul resolution of $O_{Z'}$ on $X$:

$$0 \to O_X(-\sum_i a_i) \to \cdots \to \bigoplus_i O_X(-a_i) \to O_X \to O_{Z'} \to 0$$

By a formal computation, it follows that

$$c_r(O_{Z'}) = ah^n \in H^n(\Omega^n_X)$$

where $a = (-1)^{n-1}(n-1)! \prod_i a_i \neq 0$.

On the other hand, since $L'_i$ are local complete intersections in $X$, there exist finite resolutions by vector bundles over $X$ for the sheaves $O_{L'_i}$:

$$0 \to P^* \to O_{L'_i} \to 0.$$  

We have an exact sequence,

$$0 \to O_{L'_i}(-d) \to O_{Z'} \to O_{L'_i} \to 0$$

where $d = \deg f_0$ which gives by Property 2 that

$$c(O_{Z'}) = c(O_{L'_i}) c(O_{L'_i}(-d)) \in H^*(\Omega^*_X).$$

Let $M_1$ be the subscheme defined by the vanishing of $g_0, \ldots, g_n$ in $X_{\text{red}}$. Then $\dim M_1 = n$ and since $L_1 \cap M_1 = \emptyset$, we get, $0 \to P^*_1 \otimes O_{M_1} \to 0$ is exact. Since $\dim M_1 = n = \min\{n, 2n + 1 - 2\}$, by Lemma 3, we see that $c_{\text{red}}(O_{L'_i}) = 1 + x$, where $x \in \bigoplus_{i>n} H^i(\Omega^i_{X_{\text{red}}}$. A similar argument with $L_2$ and $M_2$ (which is defined by the vanishing of $f_0, g_1, \ldots, g_n$ on $X_{\text{red}}$) gives, $c_{\text{red}}(O_{L'_i}(-d)) = 1 + y$. Thus by Property 6, $c_{\text{red}}(O_{Z'}) = 1 + z$ with $z \in \bigoplus_{i>n} H^i(\Omega^i_{X_{\text{red}}}$. In particular, we see that $c_{\text{red}}(O_{Z'}) = 0$. But, we have seen that this is the image of $ah^n$ for $a \neq 0$, $h$ the class of hyperplane section. By Lemma 2, this is a contradiction.

$$\square$$

4. Appendix

The purpose of this appendix is to prove the following theorem which is folklore, but we thought we will give a proof for completeness.

**Theorem 2.** Let $X$ be a projective scheme of dimension $r \geq 1$, $h \in H^1(X, \Omega^1_X)$ the class of a hyperplane section. Then $h^r$ in $H^r(X, \Omega^r_X)$ is not zero.
Since we plan to show some class is not zero, we will usually not worry about correctly identifying cohomology classes and allow ourselves the liberty of multiplying by non-zero constants. In other words, cohomology groups will not be canonically identified and we will allow ourselves choice of bases. We first prove a slightly stronger theorem for \( \mathbb{P}^r \).

Let \( h \in H^1(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}) \) be the class of a hyperplane. We will assume the well known facts that \( h^i \in H^i(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}) \) generates this one dimensional vector space (in particular \( h^i \neq 0 \)) and \( c_1(\mathcal{O}_{\mathbb{P}^r}(1)) \) is a non-zero multiple of \( h \).

**Theorem 3.** Let \( H \) be a hyperplane section of \( \mathbb{P}^r \) with \( r \geq 2 \). Then we have a canonical isomorphism \( \alpha : H^{r-1}(H, \Omega^1_H) \cong H^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}) \) and \( \alpha(h^{r-1}) = ch^r \), \( c \) a non-zero constant where each of the \( h \)'s is the class of an appropriate hyperplane. (If we had made the correct identifications, then \( c = 1 \)).

**Proof.** We have a canonical exact sequence,

\[
0 \to \Omega^r_{\mathbb{P}^r} \to \Omega^r_{\mathbb{P}^r}(H) \to \Omega^{r-1}_H \to 0.
\]

This gives, by taking cohomologies an isomorphism

\[
\alpha : H^{r-1}(H, \Omega^1_H) \to H^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}),
\]

using the fact that \( H^i(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r}(H)) = 0 \) for \( i = r-1, r \).

**Lemma 4.** Let \( C \) be a non-singular projective curve and let \( L \) be an ample line bundle. Then \( l = c_1(L) \in H^1(C, \Omega^1_C) \) is not zero.

**Proof.** It is clear that we may replace \( l \) by \( nl \) for any \( n > 0 \) and thus we may assume that \( L \) is very ample. This gives, by taking two general sections of \( L \), a morphism \( f : C \to \mathbb{P}^1 \) with \( f^*(\mathcal{O}_{\mathbb{P}^1}(1)) = L \). Since

\[
l = c_1(L) = c_1(f^*(\mathcal{O}_{\mathbb{P}^1}(1))) = f^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)));
\]

it suffices to prove that \( c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \neq 0 \) which we have assumed and that \( f^* : H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \to H^1(C, \Omega^1_C) \) is injective. The second statement is obvious, since the natural map \( \Omega^1_{\mathbb{P}^1} \to f_*\Omega^1_C \) splits. \( \square \)

**Proof of Theorem 2.** Let \( Y \subset X \) be a reduced irreducible closed subvariety of dimension \( r \). We have a natural map \( H^r(X, \Omega^r_X) \to H^r(Y, \Omega^r_Y) \). Thus it suffices to prove the theorem for \( Y \), since \( h^r \) goes to \( h^r \). Thus we may assume that \( X \) is integral. Similarly, we may replace \( X \) by its normalization and thus assume that \( X \) is normal. Proof is by induction on \( r \) where the case \( r = 1 \) is treated in lemma 4.
For the induction step we proceed as follows. If $h$ is the class of the ample line bundle $H$, we may clearly replace $H$ by $nH, n > 0$. Thus we may assume the following. $H^r(X, \Omega^r_X(H)) = 0$ for $i = r - 1, r$, since $r \geq 2$. Further, we have a section $Y \in |H|$ which is integral and normal and the multiplication map $Y : \Omega^r_X \to \Omega^r_X(H)$ is injective. Let $E$ be the cokernel of this map. We may also assume that we have a finite map $f : X \to \mathbb{P}^r$ such that $f^*(\mathcal{O}_{\mathbb{P}^r}(1)) = H$ and a section $s \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ such that $f^*s$ corresponds to $Y$. Let us denote the hyperplane $s = 0$ by $L$. By our assumption, we see that the map $H^{r-1}(Y, E) \to H^r(X, \Omega^r_X)$ is an isomorphism. We also see that on the smooth points of $Y$, $E \cong \Omega^{r-1}_Y$. This says that the double dual of $E$ and $\Omega^{r-1}_Y$ are isomorphic. We denote the double dual by $F$. Thus we have maps $E \to F$ and $\Omega^{r-1}_Y \to F$ which are isomorphisms on the open subset of smooth points. Since the codimension of the singular locus is at least 2, we see that

$$H^{r-1}(Y, E) \cong H^{r-1}(Y, F) \cong H^{r-1}(Y, \Omega^{r-1}_Y).$$

Using $f$, we have a commutative diagram,

$$\begin{align*}
H^{r-1}(L, \Omega^{r-1}_L) &\xrightarrow{\cong} H^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}) \\
\downarrow f^* & \quad \downarrow f^* \\
H^{r-1}(Y, E) &\xrightarrow{\cong} H^r(X, \Omega^r_X)
\end{align*}$$

We have the natural map $H^{r-1}(L, \Omega^{r-1}_L) \xrightarrow{L^*} H^{r-1}(Y, \Omega^{r-1}_Y)$ and the class of $h^{r-1}$ goes to a non-zero element by induction. But the latter group is isomorphic to $H^{r-1}(Y, E)$ and thus $h^{r-1}$ goes to a non-zero element in this group and then by the above isomorphism, its image in $H^r(X, \Omega^r_X)$ is non-zero. Now, following $h^{r-1}$ via the other branch of the commutative diagram, we see that $h^r \neq 0$ in $H^r(X, \Omega^r_X)$ by theorem 3.

□

References
