

ON THE GEOMETRY OF GENERALISED QUADRICS

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1. INTRODUCTION

We work over the field of complex numbers \mathbb{C} .

Let $\{f_0, \dots, f_n; g_0, \dots, g_n\}$ be a regular sequence in \mathbb{P}^{2n+1} and suppose that $Q = \sum_{i=0}^n f_i g_i$ is a homogeneous polynomial. We shall refer to the hypersurface X defined by Q as a *generalised quadric*. In this note, we prove that generalised quadrics in \mathbb{P}^{2n+1} for $n \geq 1$ are reduced.

In characteristic $p > 0$, it is easy to construct generalised quadrics which are non-reduced. By exploiting this fact, low rank vector bundles were constructed on \mathbb{P}^4 and \mathbb{P}^5 in [4]. Furthermore, in characteristic 0, reducible generalised quadrics exist in \mathbb{P}^3 ; for instance, the hypersurface defined by $X^2 Y^2 - Z^2 U^2 = 0$, where X, Y, Z, U are the coordinates of \mathbb{P}^3 , is such a generalised quadric. We do not know any examples of reducible generalised quadrics in higher dimensional projective spaces. However, the question of non-reducedness is settled by our main theorem.

2. ATIYAH CLASS AND CHERN CLASSES OF VECTOR BUNDLES OVER SCHEMES

Let X be any scheme and E be any vector bundle on X . We recall that the *Atiyah class* $at(E)$ (see [1]) of the vector bundle E is the natural extension class

$$0 \rightarrow \Omega_X^1 \otimes E \rightarrow \mathcal{P}(E) \rightarrow E \rightarrow 0$$

where $\mathcal{P}(E)$ is the *principal parts bundle* of E . Thus $at(E)$ is an element of the cohomology group $H^1(X, \Omega_X^1 \otimes \mathcal{E}nd(E))$. Starting with this class, one can define *Chern-Hodge classes* $c_i(E) \in H^i(X, \Omega_X^i)$ as follows (see [3] or for a simpler exposition see [5]).

Consider the composition

$$(\Omega_X^1 \otimes \mathcal{E}nd(E))^{\otimes m} \rightarrow (\Omega_X^1)^{\otimes m} \otimes \mathcal{E}nd(E^{\otimes m}) \rightarrow \Omega_X^m \otimes \mathcal{E}nd(\wedge^m E) \rightarrow \Omega_X^m$$

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where the last map is induced by the trace map $\mathcal{E}nd(\wedge^m E) \rightarrow \mathcal{O}_X$. We then define (upto a constant non-zero factor) the Chern-Hodge classes from the composite map below:

$$\begin{aligned} \mathrm{H}^1(X, \Omega_X^1 \otimes \mathcal{E}nd(E))^{\otimes m} &\rightarrow \mathrm{H}^m(X, (\Omega_X^1 \otimes \mathcal{E}nd(E))^{\otimes m}) \rightarrow \mathrm{H}^m(X, \Omega_X^m) \\ \mathrm{at}(E)^{\otimes m} &\mapsto \mathrm{at}(E) \cup \cdots \cup \mathrm{at}(E) \mapsto c_m(E) \end{aligned}$$

By convention, $c_0(E) = 1 \in \mathrm{H}^0(X, \mathcal{O}_X)$. Furthermore, we let $c(E) = \sum c_i(E)$ which is an invertible element in the graded commutative ring $\bigoplus_i \mathrm{H}^i(X, \Omega_X^i)$.

Now let X be any scheme and let \mathcal{F} be a coherent sheaf on X which has a finite resolution by vector bundles

$$0 \rightarrow \mathrm{P}_X^\bullet \rightarrow \mathcal{F} \rightarrow 0$$

Definition 1. $c(\mathcal{F}) = c(\mathrm{P}_X^\bullet) := \prod_k c(\mathrm{P}_X^k)^{(-1)^k} \in \bigoplus \mathrm{H}^i(X, \Omega_X^i)$.

We recall some basic properties of the Chern-Hodge classes. Let $\mathcal{P}(X)$ be the set of all sheaves on X which have a finite resolution by vector bundles.

Properties:

- (1) For any sheaf $\mathcal{F} \in \mathcal{P}(X)$, $c(\mathcal{F})$ is independent of the resolution.
- (2) For any short exact sequence of sheaves in $\mathcal{P}(X)$

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

$$c(\mathcal{F}) = c(\mathcal{F}') c(\mathcal{F}'').$$

- (3) For any morphism $f : Y \rightarrow X$, there is a natural ring homomorphism $f^* : \bigoplus_i \mathrm{H}^i(X, \Omega_X^i) \rightarrow \bigoplus_i \mathrm{H}^i(Y, \Omega_Y^i)$ under which if E is a bundle on X , then $f^* c(E) = c(f^* E)$.
- (4) For any bundle E and a line bundle \mathcal{L} , we have

$$c_r(E \otimes \mathcal{L}) = \sum_{i=0}^r c_i(E) c_1(\mathcal{L}^{r-i})$$

- (5) If $\mathcal{F} \in \mathcal{P}(X)$ and

$$0 \rightarrow \mathrm{P}_X^\bullet \rightarrow \mathcal{F} \rightarrow 0$$

is a finite resolution by vector bundles and if $f : Y \rightarrow X$ is any morphism, we can define $c^Y(\mathcal{F}) \in \mathrm{H}^\bullet(Y, \Omega_Y^\bullet)$ as $c(f^* \mathrm{P}_X^\bullet)$. In general, this is not $c(f^* \mathcal{F})$, since this sheaf may not have a finite resolution by vector bundles on Y . These coincide if

$$0 \rightarrow f^* \mathrm{P}_X^\bullet \rightarrow f^* \mathcal{F} \rightarrow 0$$

remains exact and thus in this case $c^Y(\mathcal{F}) = c(f^* \mathcal{F})$.

(6) For any short exact sequence of sheaves in $\mathcal{P}(X)$

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

on X and a morphism $f : Y \rightarrow X$, $c^Y(\mathcal{F}) = c^Y(\mathcal{F}') c^Y(\mathcal{F}'')$.

The following lemma, which is the key lemma, is essentially due to Gruson et.al [2].

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Consider the restriction maps*

$$H^i(\Omega_{\mathbb{P}^n}^i) \xrightarrow{\alpha} H^i(\Omega_{X_{\text{red}}}^i)$$

and

$$H^i(\Omega_X^i) \xrightarrow{\beta} H^i(\Omega_{X_{\text{red}}}^i).$$

Then $\text{Im } \beta = \text{Im } \alpha$ for $1 \leq i < n - 1$.

Proof. Since α factors through $H^i(\Omega_X^i)$, we only need to show that $\text{Im } \beta \subset \text{Im } \alpha$. Since X is irreducible, we may assume that X is defined by a homogeneous polynomial f^m , $m > 1$ with f irreducible and so X_{red} is given by the vanishing of f . We consider the exact sequence

$$\mathcal{O}_X(-\deg(f^m)) \xrightarrow{d(f^m)} \Omega_{\mathbb{P}^n}^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1 \rightarrow 0$$

Restricting it to X_{red} , we get

$$(1) \quad \Omega_{\mathbb{P}^n}^1 \otimes \mathcal{O}_{X_{\text{red}}} \cong \Omega_X^1 \otimes \mathcal{O}_{X_{\text{red}}}$$

This implies similar isomorphisms,

$$\Omega_{\mathbb{P}^n}^i \otimes \mathcal{O}_{X_{\text{red}}} \cong \Omega_X^i \otimes \mathcal{O}_{X_{\text{red}}},$$

for all i .

Since α factors through $H^i(\Omega_{\mathbb{P}^n}^i \otimes \mathcal{O}_{X_{\text{red}}})$ and similarly β factors through $H^i(\Omega_X^i \otimes \mathcal{O}_{X_{\text{red}}})$, it suffices to prove that the map

$$H^i(\Omega_{\mathbb{P}^n}^i) \xrightarrow{\delta} H^i(\Omega_{\mathbb{P}^n}^i \otimes \mathcal{O}_{X_{\text{red}}})$$

is onto by the isomorphism (1) above. We have an exact sequence,

$$0 \rightarrow \Omega_{\mathbb{P}^n}^i(-d) \rightarrow \Omega_{\mathbb{P}^n}^i \rightarrow \Omega_{\mathbb{P}^n}^i \otimes \mathcal{O}_{X_{\text{red}}} \rightarrow 0,$$

where $d = \deg f$. Taking cohomologies and noting that $H^j(\Omega_{\mathbb{P}^n}^i(-d)) = 0$ for $j = i, i + 1$, since $1 \leq i < n - 1$, we see that δ is an isomorphism. \square

Lemma 2. *Let $M \subset \mathbb{P}^n$ be a closed subscheme of dimension r . Then the natural map,*

$$\gamma : H^i(\Omega_{\mathbb{P}^n}^i) \rightarrow H^i(\Omega_M^i)$$

is injective for $0 \leq i \leq r$.

Proof. If $h \in H^1(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^1)$ is the class of the hyperplane section, then $H^i(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^i)$ is a one dimensional vector space generated by h^i . Thus, it suffices to show that its image in $H^i(M, \Omega_M^i)$ is non-zero. If it is zero for some $i < r$, then $h^r = h^i h^{r-i} = 0 \in H^r(M, \Omega_M^r)$. A proof of the well known fact that $h^r \neq 0$ is sketched in the Appendix. \square

Lemma 3. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Let \mathcal{F} be a coherent sheaf on X with a resolution $0 \rightarrow P_X^\bullet \rightarrow \mathcal{F} \rightarrow 0$ by vector bundles on X such that $0 \rightarrow P_X^\bullet \otimes \mathcal{O}_M \rightarrow 0$ is exact where $M \subset X_{\text{red}}$ and $\dim M = r$. Then $0 = c_i^{X_{\text{red}}}(\mathcal{F}) \in H^i(X, \Omega_{X_{\text{red}}}^i)$ for $1 \leq i \leq \min\{r, n-2\}$.*

Proof. Since $0 \rightarrow P_X^\bullet \otimes \mathcal{O}_M \rightarrow 0$ is exact, $c^M(\mathcal{F}) = 1$ by Property 5. From Lemma 1 above, it follows that $\forall 1 \leq i \leq \min\{r, n-2\}$, there exist classes $t_i \in H^i(\Omega_{\mathbb{P}^n}^i)$ such that

$$\beta(c_i(\mathcal{F})) = \alpha(t_i).$$

Let $\theta : H^i(\Omega_{X_{\text{red}}}^i) \rightarrow H^i(\Omega_M^i)$ be the natural map. Then $\theta\beta(c_i(\mathcal{F})) = c_i^M(\mathcal{F}) = 0$ for $i > 0$. Thus, $\theta\alpha(t_i) = 0$. But, $\theta\alpha = \gamma$ and by Lemma 2, we get that $t_i = 0$ for $1 \leq i \leq \min\{r, n-2\}$ and thus

$$c_i^{X_{\text{red}}}(\mathcal{F}) = \beta c_i(\mathcal{F}) = 0$$

for $1 \leq i \leq \min\{r, n-2\}$. \square

3. GENERALISED QUADRICS

In this section, we apply the results of the previous section to show that generalised quadrics in \mathbb{P}^{2n+1} for $n \geq 1$ are reduced.

Let $Q \subset \mathbb{P}^{2n+1}$ denote the generalised quadric given by the equation $\sum_{i=0}^n f_i g_i = 0$. Let

$$Z := Q \cap (f_1 = \cdots = f_n = 0)$$

$$L_1 := Q \cap (f_0 = \cdots = f_n = 0)$$

$$L_2 := Q \cap (g_0 = f_1 = \cdots = f_n = 0)$$

Then $Z = L_1 \cup L_2$ and we have an exact sequence

$$0 \rightarrow \mathcal{O}_{L_2}(-\deg f_0) \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{L_1} \rightarrow 0$$

Furthermore, Z is a complete intersection of n ample divisors on Q , L_i for $i = 1, 2$ are local complete intersection subschemes in Q of codimension (and dimension) n .

Theorem 1. *The generalised quadric Q is reduced.*

Proof. If Q is not reduced, let X be an irreducible component of Q which is not reduced and let X_{red} denote the subscheme X with the reduced structure. Thus $\sum f_i g_i = f^r f'$ with f an irreducible polynomial, $r > 1$ where $f^r = 0$ defines X and $f = 0$ defines X_{red} .

Let $Z' = Z \cap X$, $L'_i = L_i \cap X$. It is easy to see that Z' is a complete intersection in X by $f_i, i > 0$. We consider the Koszul resolution of $\mathcal{O}_{Z'}$ on X :

$$0 \rightarrow \mathcal{O}_X(-\sum_i a_i) \rightarrow \cdots \rightarrow \oplus_i \mathcal{O}_X(-a_i) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{Z'} \rightarrow 0$$

By a formal computation, it follows that

$$c_n(\mathcal{O}_{Z'}) = ah^n \in H^n(\Omega_X^n)$$

where $a = (-1)^{n-1}(n-1)! (\prod_i a_i) \neq 0$.

On the other hand, since L'_i are local complete intersections in X , there exist finite resolutions by vector bundles over X for the sheaves $\mathcal{O}_{L'_i}$:

$$0 \rightarrow P_i^\bullet \rightarrow \mathcal{O}_{L'_i} \rightarrow 0.$$

We have an exact sequence,

$$0 \rightarrow \mathcal{O}_{L'_2}(-d) \rightarrow \mathcal{O}_{Z'} \rightarrow \mathcal{O}_{L'_1} \rightarrow 0$$

where $d = \deg f_0$ which gives by Property 2 that

$$c(\mathcal{O}_{Z'}) = c(\mathcal{O}_{L'_1}) c(\mathcal{O}_{L'_2}(-d)) \in H^\bullet(\Omega_X^\bullet).$$

Let M_1 be the subscheme defined by the vanishing of g_0, \dots, g_n in X_{red} . Then $\dim M_1 = n$ and since $L_1 \cap M_1 = \emptyset$, we get, $0 \rightarrow P_1^\bullet \otimes_{\mathcal{O}_{M_1}} \rightarrow 0$ is exact. Since $\dim M_1 = n = \min\{n, 2n+1-2\}$, by Lemma 3, we see that $c^{X_{\text{red}}}(\mathcal{O}_{L'_1}) = 1+x$, where $x \in \oplus_{i>n} H^i(\Omega_{X_{\text{red}}}^i)$. A similar argument with L_2 and M_2 (which is defined by the vanishing of f_0, g_1, \dots, g_n on X_{red}) gives, $c^{X_{\text{red}}}(\mathcal{O}_{L'_2}(-d)) = 1+y$. Thus by Property 6, $c^{X_{\text{red}}}(\mathcal{O}_{Z'}) = 1+z$ with $z \in \oplus_{i>n} H^i(\Omega_{X_{\text{red}}}^i)$. In particular, we see that $c_n^{X_{\text{red}}}(\mathcal{O}_{Z'}) = 0$. But, we have seen that this is the image of ah^n for $a \neq 0$, h the class of hyperplane section. By Lemma 2, this is a contradiction. \square

4. APPENDIX

The purpose of this appendix is to prove the following theorem which is folklore, but we thought we will give a proof for completeness.

Theorem 2. *Let X be a projective scheme of dimension $r \geq 1$, $h \in H^1(X, \Omega_X^1)$ the class of a hyperplane section. Then h^r in $H^r(X, \Omega_X^r)$ is not zero.*

Since we plan to show some class is not zero, we will usually not worry about correctly identifying cohomology classes and allow ourselves the liberty of multiplying by non-zero constants. In other words, cohomology groups will not be canonically identified and we will allow ourselves choice of bases. We first prove a slightly stronger theorem for \mathbb{P}^r .

Let $h \in H^1(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^1)$ be the class of a hyperplane. We will assume the well known facts that $h^i \in H^i(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^i)$ generates this one dimensional vector space (in particular $h^i \neq 0$) and $c_1(\mathcal{O}_{\mathbb{P}^r}(1))$ is a non-zero multiple of h .

Theorem 3. *Let H be a hyperplane section of \mathbb{P}^r with $r \geq 2$. Then we have a canonical isomorphism $\alpha : H^{r-1}(H, \Omega_H^{r-1}) \rightarrow H^r(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^r)$ and $\alpha(h^{r-1}) = ch^r$, c a non-zero constant where each of the h 's is the class of an appropriate hyperplane. (If we had made the correct identifications, then $c = 1$).*

Proof. We have a canonical exact sequence,

$$0 \rightarrow \Omega_{\mathbb{P}^r}^r \rightarrow \Omega_{\mathbb{P}^r}^r(H) \rightarrow \Omega_H^{r-1} \rightarrow 0.$$

This gives, by taking cohomologies an isomorphism

$$\alpha : H^{r-1}(H, \Omega_H^{r-1}) \rightarrow H^r(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^r),$$

using the fact that $H^i(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^r(H)) = 0$ for $i = r - 1, r$. □

Lemma 4. *Let C be a non-singular projective curve and let L be an ample line bundle. Then $l = c_1(L) \in H^1(C, \Omega_C^1)$ is not zero.*

Proof. It is clear that we may replace l by nl for any $n > 0$ and thus we may assume that L is very ample. This gives, by taking two general sections of L , a morphism $f : C \rightarrow \mathbb{P}^1$ with $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) = L$. Since

$$l = c_1(L) = c_1(f^*(\mathcal{O}_{\mathbb{P}^1}(1))) = f^*(c_1(\mathcal{O}_{\mathbb{P}^1}(1))),$$

it suffices to prove that $c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \neq 0$ which we have assumed and that $f^* : H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \rightarrow H^1(C, \Omega_C^1)$ is injective. The second statement is obvious, since the natural map $\Omega_{\mathbb{P}^1}^1 \rightarrow f_*\Omega_C^1$ splits. □

Proof of Theorem 2. Let $Y \subset X$ be a reduced irreducible closed subvariety of dimension r . We have a natural map $H^r(X, \Omega_X^r) \rightarrow H^r(Y, \Omega_Y^r)$. Thus it suffices to prove the theorem for Y , since h^r goes to h^r . Thus we may assume that X is integral. Similarly, we may replace X by its normalization and thus assume that X is normal. Proof is by induction on r where the case $r = 1$ is treated in lemma 4.

For the induction step we proceed as follows. If h is the class of the ample line bundle H , we may clearly replace H by $nH, n > 0$. Thus we may assume the following. $H^i(X, \Omega_X^r(H)) = 0$ for $i = r - 1, r$, since $r \geq 2$. Further, we have a section $Y \in |H|$ which is integral and normal and the multiplication map $Y : \Omega_X^r \rightarrow \Omega_X^r(H)$ is injective. Let \mathcal{E} be the cokernel of this map. We may also assume that we have a finite map $f : X \rightarrow \mathbb{P}^r$ such that $f^*(\mathcal{O}_{\mathbb{P}^r}(1)) = H$ and a section $s \in H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ such that f^*s corresponds to Y . Let us denote the hyperplane $s = 0$ by L . By our assumption, we see that the map $H^{r-1}(Y, \mathcal{E}) \rightarrow H^r(X, \Omega_X^r)$ is an isomorphism. We also see that on the smooth points of Y , $\mathcal{E} \cong \Omega_Y^{r-1}$. This says that the double dual of \mathcal{E} and Ω_Y^{r-1} are isomorphic. We denote the double dual by \mathcal{F} . Thus we have maps $\mathcal{E} \rightarrow \mathcal{F}$ and $\Omega_Y^{r-1} \rightarrow \mathcal{F}$ which are isomorphisms on the open subset of smooth points. Since the codimension of the singular locus is at least 2, we see that

$$H^{r-1}(Y, \mathcal{E}) \cong H^{r-1}(Y, \mathcal{F}) \cong H^{r-1}(Y, \Omega_Y^{r-1}).$$

Using f , we have a commutative diagram,

$$\begin{array}{ccc} H^{r-1}(L, \Omega_L^{r-1}) & \xrightarrow{\cong} & H^r(\mathbb{P}^r, \Omega_{\mathbb{P}^r}^r) \\ \downarrow f^* & & \downarrow f^* \\ H^{r-1}(Y, \mathcal{E}) & \xrightarrow{\cong} & H^r(X, \Omega_X^r) \end{array}$$

We have the natural map $H^{r-1}(L, \Omega_L^{r-1}) \xrightarrow{f^*} H^{r-1}(Y, \Omega_Y^{r-1})$ and the class of h^{r-1} goes to a non-zero element by induction. But the latter group is isomorphic to $H^{r-1}(Y, \mathcal{E})$ and thus h^{r-1} goes to a non-zero element in this group and then by the above isomorphism, its image in $H^r(X, \Omega_X^r)$ is non-zero. Now, following h^{r-1} via the other branch of the commutative diagram, we see that $h^r \neq 0$ in $H^r(X, \Omega_X^r)$ by theorem 3. \square

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