ON A THEOREM OF SESHADRI

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1. Introduction

The aim of this article is to describe a fundamental result proved by Professor C. S. Seshadri in 1958 [Ses58], its genesis and subsequent developements. As the reader will see, this first step taken by Seshadri has helped others to leap forward and generate a tremendous amount of first rate mathematics in the last few decades.

The article is organised as follows. In the first section, we discuss some basic notions so that one can better understand the problem that Seshadri dealt with. This problem was posed by J. -P. Serre [Ser55, page 243] and is known as *Serre's Problem*. For the uninitiated, this discussion might seem a little abstruse, but we will end the section by showing that the problem is equivalent to a very easily stated and clearly fundamental question. One hopes that this will serve as motivation for a reader with some basic mathematical background. In this section, we will sketch a proof of Seshadri's theorem. In the second section, we will discuss some of the later developements culminating with the complete solution to Serre's Problem. We will sketch the proof of a key step in the complete solution, due to G. Horrocks. In the last section, we will discuss some of the applications to these results and how these results have helped in forging new branches of mathematics and grown into a fertile ground for impressive later developements. I hope that the applications we discuss will help the reader put the problem of Serre and Seshadri's theorem in context.

Considering the amount of literature on the subject, the article will be far from exhaustive. There are several excellent accounts of the problem and progress made at various stages by a number of mathematicians and the reader will be able to find them in the references at the end of the aricle. Almost no proofs are given, but one hopes that the references at the end will lead the reader to a more satisfying mathematical experience.

I would like to warn the reader that the work of Seshadri we discuss is but a small piece of his enormous output. This is one of his earliest works, albeit an extremely important one. He, as a young mathematician in Paris, came under the influence of several important mathematicians, like C. Chevalley, J.-P. Serre and A. Grothendieck, just to name a few. He returned to India as one of the first modern Indian Algebraic Geometers and essentially influenced almost all the senior Algebraic Geometers from Tata Institute. The nucleus that he built has blossomed into one of the strongest schools of Geometers in the world today. His achievements in the last several decades on Moduli of Vector bundles on curves, Geometric Invariant Theory, Representation theory and Schubert Calculus are so impressive as to almost diminish his 'baby work' that we will be dicussing in this article. But, I hope to convince the reader of the importance of this earlier work as best as I can.

Work partially supported by NSF grants.

I thank Professors R. Bhatia and C. Musili for asking me to write this article on the work of Professor Seshadri. I thank Professor Seshadri for being my inspiration and to a great extent my motivation at the Tata Institute of Fundamental Research, while I was a student there.

2. Serre's Problem

In general all rings we consider will be commutative, Noetherian[1] with a multiplicative identity 1, unless otherwise mentioned. We assume that the reader has some basic knowledge about rings and modules. Some good references are [AM69], [Ser00], [Mat80] and [Lan02], where the reader will find all the notions that we will deal with. For example, we will asssume that the reader knows that if A is a ring, M, N are modules, then we can construct a new module, $M \oplus N$, the direct sum. Similarly, taking a leaf from the theory of vector spaces, we have *free modules*, $F = \oplus_{i=1}^{n} Ae_i$, where e_i's are the basis for the free module F of rank n. A subset $S \subset A$ is called a *multiplicatively closed subset*, if $0 \notin S$ and for any $a, b \in S$, $ab \in S$. The idea is that, given such an S, we can construct a new ring, $S^{-1}A$, called the *localisation* of A at S, by considering (formally) fractions of the form a/s, with $a \in A, s \in S$. Similarly, given an A-module M, we can consider ratios m/s with $m \in M$ and $s \in S$ to construct an $S^{-1}A$-module $S^{-1}M$, called localisation of M at S. Two of the most important kinds of multiplicatively closed sets are:

(1) Let $f \in A$ be not nilpotent. Then take $S = \{1, f, f^2, f^3, \ldots\}$. In this case, one often writes A_f instead of $S^{-1}A$.
(2) Let $\mathfrak{p} \subset A$ be a prime ideal and let $S = A - \mathfrak{p}$. In this case one writes $A_{\mathfrak{p}}$ instead of $S^{-1}A$.

Now we come to a basic definition:

Definition 1. *Let A be a ring and P a module over A. Then P is a projective module if there exists a module Q such that $P \oplus Q \cong F$, where F is a free module.*

Note: We will always assume that the modules we deal with are finitely generated.

There are two important characterisations of projective modules P over A.

(1) If $\phi : M \to P$ is a surjection of A-modules, then there exists an A-module homomorphism, $\psi : P \to M$ such that $\phi \circ \psi = \mathrm{Id}_P$, where Id_P is the identity map from P to itself. This is usually expressed as any surjection to a projective module *splits.*
(2) P is projective if and only if $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p}$ of A if and only if $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m}$ of A. This condition is often expressed as a module being *locally free.* So, locally, the structure of a projective module is as simple as it can be. The real issues are global.

Next, we give some simple but typical examples of projective modules.

(1) Let A be a principal ideal domain. Then all finitely generated projective modules (torsion-free is enough) are free.
(2) Let $\mathcal{O}$ be the ring of integers in a number field and let h be the class number. Then for any given rank $n > 0$, there exist exactly h non-isomorphic projective modules of rank n. In particular, if $h > 1$, there exist non-free projective modules of any rank $n > 0$.

[1] This just means that all ideals are finitely generated.

(3) Let $A = \mathbb{C}[x,y]/(y^2 - x^3 - x)$, the ring of algebraic functions on the open *elliptic curve*, $y^2 - x^3 - x = 0$. Let $I = (x, y)$, the ideal generated by x, y. Then, I is a projective module over A of rank one, which is not free.

(4) Let T be the tangent bundle of the real sphere. This means we look at the ring $A = \mathbb{R}[x,y,z]/(x^2 + y^2 + z^2 - 1)$ and the module, $T = Ae_1 \oplus Ae_2 \oplus Ae_3/A \cdot (xe_1 + ye_2 + ze_3)$. Then T is a rank two projective module over A which is not free.

In his paper *Faisceaux algébriques cohérents* [Ser55, page 243], J.-P. Serre raised the following problem:

Serre's Problem: Let P be a projective module over $R = k[x_1, x_2, \ldots x_n]$, polynomial ring in n variables over a field k. Then is P free?

By the first example above, this is indeed true for $R = k[x]$, since then R is a principal ideal domain. When Serre raised this question, this was the only case in which the answer to the above problem was known. The locally free nature of projective modules imply that they can be identified with *vector bundles* on suitable toplological spaces or more precisely on certain affine varieties. For example, a projective module over $\mathbb{C}[x_1, x_2, \ldots, x_n]$ would correspond to a vector bundle on $\mathbb{C}^n$. From topology and complex (real) analysis, there is evidence that such vector bundles are trivial, which translates in our language to freeness of the projective module. So there were some excellent reasons for the validity of Serre's Problem, though the techniques then known for topological or holomorphic categories did not easily translate into the algebraic situation. Now we are ready to state Seshadri's theorem.

Theorem 1 (Seshadri). *Projective modules over $k[x,y]$, polynomial ring in two variables over a field k, are free.*

Remark: Seshadri, in fact, proved that projective modules over $A[x]$ are free where A is a principal ideal domain and later generalised this to the case where A may be also a Dedekind domain which occurs as the ring of functions on a non-singular curve over an algebraically closed field [Ses59]. Later H. Bass generalised this to the case of an arbitrary Dedekind domain [Bas62]. However as I have noted in an example before, projective modules are not free in this situation, but they are always the direct sum of a free module and a projective module of rank one.

I briefly sketch Seshadri's proof, with some modifications, though true to it in spirit. Let P be a projective module over $R = k[x,y]$. Consider S, the set of all non-zero polynomials in x over k. Then S is a multiplicatively closed subset and $S^{-1}R = k(x)[y]$, which is a polynomial ring in y over the field $k(x)$ of rational functions in x. Thus, $S^{-1}P$, which is a projective module over $S^{-1}R$, by the one variable case is a free $S^{-1}R$-module. By choosing a basis, one easily sees that we can trap our projective module between a free module F and fF where $0 \neq f \in k[x]$. Using all such possibilities one can reduce (with a little effort) to the case when f is in fact an irreducible polynomial. Again, by suitable filtering, one will be reduced to the case where P is given as follows.

$$0 \to P \to F \to R/f \to 0.$$

That is, P is the kernel of some surjective map $F \to R/f$. (Seshadri did not do this via the above filtration, but directly appealed to the fact that any $n \times n$ matrix with determinant one over a polynomial ring in one variable, or more genrally a principal ideal domain is *elemenatay*, that is, obtained by row and column operations. We chose to do it differently, since then the arguments are more transparent.) Choosing some basis e_i of F, the images of e_i go to say, $g_i \in R/f = (k[x]/f)[y]$. Since the map is onto, we see that the g_i's must have their greatest common divisor 1. Then, by division algorithm, we can elementarily transform the vector $(g_1, \dots g_n)$ to $(1, 0, \dots, 0)$. Let me explain this a little further. Choose $g_i \neq 0$, whose degree (rememeber that all the g_i's are polynomials in y with coefficients in the field $k[x]/f$) is the least among the $g_j \neq 0$. Then, division algorithm says that we can subtract a suitable multiple of g_i from g_j and reduce its degree. But, this operation can be carried out with our e_j's. If we want to get $g_j - tg_i$, then we can find some $u \in k[x, y]$ which maps to t (since the map $k[x, y] \to k[x, y]/f$ is surjective) and replace e_j by $e_j - ue_i$. This has the effect of just choosing a different basis for F, namely, $e'_i = e_i$ for $i \neq j$ and $e'_j = e_j - ue_i$. So, doing this finitely many times, with the new basis, we have $e_1 \mapsto 1 \in R/f$ and the remaining e_i's go to zero. At this point, the kernel P can easily seen to be free with the basis $fe_1, e_2, \dots e_n$. This finishes the proof of Seshadri's theorem.

Next, I want to convert Serre's problem into a more elementary statement, so that its basic nature can be better appreciated.

If M is any module over $R = k[x_1, \dots, x_n]$, then we have a presentation,

$$F_1 \xrightarrow{\phi} F_0 \to M \to 0,$$

where F_i's are free and ϕ is a matrix over R. This is just a convenient way of expressing that if we choose some onto map from $F_0 \to M$ (which is possible, since M is finitely generated), we can find a finitely generated free module F_1 mapping onto the kernel of the map $F_0 \to M$, since the ring is Noetherian and thus submodules of a finitely generated module are finitely generated. So, M essentially codifies the same information as ϕ. Now, we can add an extra variable, x_0 and homogenise ϕ with respect to x_0. So, we get a matrix Φ, whose entries are homogeneous polynomials of same degree over $S = k[x_0, x_1, \dots, x_n]$ and when we substitute $x_0 = 1$, we get our ϕ back. We can consider Φ as a map from $G_1 \to G_0$, where G_i's are free modules over S and the rank of G_i is the rank of F_i. Calling N the cokernel of Φ, we have,

$$G_1 \xrightarrow{\Phi} F_0 \to N \to 0.$$

What is the advantage of doing all this? Well, we have converted our ϕ into a Φ, which has homogeneous polynomials as entries. This makes N into a *graded* module over S. This is helpful, thanks to a theorem of Hilbert [Hil65]. The Hilbert Syzygy Theorem allows us to write a *resolution*,

$$0 \to G_d \to G_{d-1} \to \cdots \to G_1 \to G_0 \to N \to 0,$$

where G_i's are free modules and $d \leq n + 1$. This means that G_k maps onto the kernel of the map $G_{k-1} \to G_{k-2}$ for all k. Now, put $x_0 = 1$. The G_i's when we put $x_0 = 1$ becomes F_i, free modules over R and thus we get a resolution,

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Further, if we assume that M is projective, then by 'splitting' we see that $M \oplus F \cong G$ for free modules F, G over R, where

$$F = F_1 \oplus F_3 \oplus \cdots, G = F_0 \oplus F_2 \oplus \cdots.$$

So, we have proved,

Theorem 2 (Serre). *If P is a projective module over $R = k[x_1, \dots, x_n]$, then there exists free modules F, G such that $P \oplus F \cong G$.*

This theorem, attributed to Serre appears in [Ser58] This is usually referred to as P being *stably free*. In the language of K-theory, this is also stated as $K_0(k[x_1, \dots, x_n] \cong \mathbb{Z}$. Thus to settle Serre's problem, by inducting on the rank of F above, it suffices to show that if $P \oplus R \cong R^{m+1}$ (where we have used the notation R^{m+1} to denote a free module of rank $m+1$), then $P \cong R^m$. If $P \oplus R \cong R^{m+1}$, then the element $(0,1) \in P \oplus R$ gives a vector $(a_0, a_1, \dots, a_m) \in R^{m+1}$ via the above isomorphism, where of course $a_i \in R$. Thus

$$P = \frac{R^{m+1}}{R \cdot (a_0, a_1, \dots, a_m)}.$$

That the quotient module P is projective is easily seen to be equivalent to the existence of elements $b_i \in R$ such that $\sum a_i b_i = 1$. A row vector $v_0 = (a_0, \dots, a_m)$ is called *unimodular* if there exists elements $b_i \in R$ such that $\sum a_i b_i = 1$. If we are dealing with polynomial rings over a field or more generally, quotients of polynomial rings (called *affine algebras*), then the unimodularity also can be expressed as saying that these functions have no common zeroes in an algebraic closure of k, by Hilbert Nullstellensatz. If P is free, then lifting the basis of P to R^{m+1}, say $v_i \in R^{m+1}, 1 \leq i \leq m$, we find a new basis for R^{m+1}, namely, $v_0, v_1, \dots, v_m$. Since any two bases are related by a non-singular matrix of size $m+1 \times m+1$ over R, we see that there exists a $\sigma \in \mathrm{GL}(m+1, R)$ whose first row is $(a_0, \dots, a_m)$, where as usual, $\mathrm{GL}(m+1, R)$ denotes the group of all $m+1 \times m+1$ non-singular matrices over R. Such a row is called *completable*. So, now we can restate Serre's problem as follows:

Serre's Problem: If $(a_0, \dots, a_m)$ is a unimodular row over $R = k[x_1, \dots, x_n]$ then is it completable?

We can make one more reduction in this case. Given any non-zero polynomial $f \in k[x_1, \dots, x_n]$, after a suitable change of the variables, we may assume that f is a monic polynomial in, say x_n. Making this change for the a_i's above (since at least one of them is non-zero), we may assume that one of the a_i's is monic in x_n. Then a complete solution of Serre's problem in this form is given in the Algebra text book of Lang [Lan02] and about which I will have more to say in the next section. I mention it here only to impress upon the reader that Serre's problem is fundamental enough to be included in a standard graduate text and fortunately, one of the proofs is elementary enough to be explained in such a text.

3. Later Developements

In this section, we discuss some of the later developements on Serre's problem. There are several excellent accounts of these. For example, the reader may consult [Bas75, VS76, Fer77].

After Seshadri settled the two variable case, naturally attention turned to three or more variables. Around the time of Seshadri's result, Serre had already proved that if P is a projective module over $R = k[x_1, \dots, x_n]$, then $P = Q \oplus F$, where Q is a projective module of rank at most n and F is free. In other words, one really need to look at only projective modules of rank at most n. (This statement is more general and is true for any ring R, not necessarily polynomial rings, where $n = \dim R$, the Krull dimension). G. Horrocks [Hor64] proved that projective modules are free over $R[x]$, where R is a two dimensional regular local ring containing a field. M. P. Murthy [Mur66], using Seshadri's techniques improved it to the case when R may not contain a field. H. Bass had proved that projective modules of rank $> n$ are free over a polynomial ring in n variables over a field [Bas64]. A. A. Suslin and I. N. Vaserstein [VS76] proved that projective modules of rank $\geq 1 + \frac{n}{2}$ are free in the above situation. Around the same time, Moshe Roitman had partially improved the theorem of Bass and showed that projective modules over a polynomial ring in n variables of rank at least n are free, if the base field is infinite [Roi75]. In 1974, M. P. Murthy and J. Towber proved [MT74] that Serre's problem has an affirmative answer in the three variables case at least when the base field is algebraically closed. This was the first major breakthrough in fifteen years after Seshadri. More cases were done by Suslin, Vaserstein and R. G. Swan when the number of variables were at most 5, with some minor hypothesis on the field around the same time. Very soon after that, the problem was completely settled by D. Quillen [Qui76] and A. A. Suslin [Sus76] in 1976, independently and almost simultaneously. The proofs of these results are of independent interest and the reader can look it up in the original articles quoted in the references. I will illustrate only one of these results, which can be deduced using nothing more than the division algorithm of polynomials in one variable. I single this one out, since it is elementary and in some sense one of the key steps towards the complete solution of Serre's problem. Further, this is closer to the proof of Seshadri in spirit and was proved originally by G. Horrocks.

Theorem 3 (Horrocks). *Let R be a local ring with maximal ideal $\mathfrak{m}$. (This means that $\mathfrak{m} \subset R$ is the unique maximal ideal of R). Let*

$$(f_1(T), f_2(T), \ldots, f_n(T))$$

be a unimodular row over $R[T]$ with at least one of the $f_i(T)$ monic in T. Then this row is completable.

I will sketch two different proofs of this result. One, the original proof of G. Horrocks and the second, a proof by Suslin. The proof of Horrocks uses some basic facts from Algebraic Geometry and the proof of Suslin can be understood purely from the Commutative Algebra view point and uses nothing more than the division algorithm.

Proof: Let $P = R[T]^n/(f_1, \ldots, f_n)$. The fact that one of the f_i's, say f is monic implies that P_f is free over $R[T]_f$. Writing $f = T^r g(T^{-1})$, where $g(T^{-1}) \in R[T^{-1}]$ with $g(0) = 1$ and $r = \deg_T f$, we can cover $\mathbb{P}^1_R$, the projective line over R by the two open sets, $U = \text{Spec}R[T]$ and $V = \text{Spec}R[T^{-1}]_{g(T^{-1})}$. Since $U \cap V = \text{Spec}R[T]_f$, we see that if we take P on U and a free module of rank equal to the rank of P on V, then they are isomorphic on $U \cap V$, since both are now free of the same rank on $U \cap V$. Thus, we can 'patch' them up to get a vector bundle E on $\mathbb{P}^1_R$ which, when restricted to U, gives us P. Let us restrict E to the special fibre, Spec$k$ where $k = \text{Spec}R/\mathfrak{m}$, the residue field, which we call E'. Then, since k is a field, we can appeal to a theorem of Grothendieck and we see that

$$E' \cong \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_i),$$

for suitable integers a_i's. (The above result of Grothendieck can also be deduced from Seshadri's theorem, but it really can be tweaked out of the Hilbert Syzygy Theorem, thus simpler.) We may twist E' and correspondingly E by a suitable $\mathcal{O}(a)$, so that all the $a_i \geq 0$ and at least one $a_i = 0$. Then, one appeals to semi-continuity theorem [Gro61] to show that E has a nowhere vanishing section. The upshot is, that when we restrict E to U, we find that there exists a section $p \in P$ such that $Q = P/R[T]p$ is a projective module over $R[T]$ and Q is free when we invert a monic polynomial in T. So, Q satisfies the conditions that P satisfied, but the rank of Q is one less than P. Since $P = R[T] \oplus Q$, an easy induction on the rank of P finishes the proof.

Now, let me give Suslin's proof. If $n = 1$ or $n = 2$, the theorem is trivial, by the unimodular property. So, we will assume that $n \geq 3$. We will make 'elementary' transformations on the vector $(f_1, \dots, f_n)$ and convert it to the vector, $(1, 0, \dots, 0)$, which will finish the proof. That is, we are allowed to make the following transformations to the vector $(f_1, \dots, f_n)$.

(1) We may replace f_i by $f_i + hf_j, i \neq j$ and any $h \in R[T]$.
(2) We may interchange f_i and f_j.
(3) We may multiply f_i by a unit in the ring R.

By interchanging the f_i's if necessary, we may assume that f_1 is monic in T and say of degree d. If $d = 0$, then $f_1 = 1$, being monic. Then, we can subtract $f_j \cdot f_1$ from f_j for $j > 1$ and get all the other entries to be zero. So, we are done. So, let us assume that $d > 0$. By division algorithm, we may arrange that $\deg f_j < d, j > 1$ (which just involves subtracting suitable multiples of f_1 from the f_j's). Now, go modulo the maximal ideal $\mathfrak{m} \subset R$. Then the unimodularity implies, since $f_1(T)$ is not a invertible in $k[T]$, where $k = R/\mathfrak{m}$ being of positive degree, at least one of the $f_j(T) \neq 0$ modulo $\mathfrak{m}$. Again, without loss of generality, we may assume that $f_2(T)$ is not congruent to zero modulo $\mathfrak{m}$. If we write $f_2(T)$ as $a_rT^r + \cdots + a_0$, this says that some a_i is a unit in R. Notice that $r < d$. Let i be so chosen that a_i is a unit and i is maximum. So, $a_j \in \mathfrak{m}$ for all $i < j \leq r$. We may further assume that the leading coefficient of $f_j(T)$ are all in the maximal ideal, otherwise, we have a situation in one of the f_j's is monic (upto a unit in R) of degree less than d and we can start from here.

Now, multiply f_2 by T^{d-1-i} and add to f_3, which exists, since $n \geq 3$. Then, the new f_3 looks like,

$$b_kT^k + b_{k-1}T^{k-1} + \cdots + b_0$$

and $b_l \in \mathfrak{m}$ for $l > d-1$ and $b_{d-1} \notin \mathfrak{m}$. Now, use the division algorithm using f_1 once again and we can reduce the degree of f_3 to be less than $d = \deg f_1$. But, one easily sees that, when we do this, the new f_3 we get is of degree $d-1$ and it is monic in T, upto units in R. Now, we can go back and work with f_3 instead of f_1 and an easy induction finishes the proof.

I would like to mention another important result in this direction, due to Suslin [Sus77]. The theorem is more general than what I state below. We will use the usual notation $\mathrm{SL}(m, R)$ for a ring R and an integer $m > 0$ to denote the group of all $m \times m$ matrices over R of determinant 1. Similarly, $\mathrm{E}(m, R)$ is the subgroup of $\mathrm{SL}(m, R)$ of all elementary matrices, generated by matrices corresponding to row and column operations.

Theorem 4 (Suslin). *Let $R = k[x_1, \dots, x_n]$, be a polynomial ring in n variables over a field k. Then $\mathrm{SL}(m, R) = \mathrm{E}(m, R)$, for $m \geq 3$.*

Let me conclude this section by stating the main theorem of Quillen in [Qui76], which is very general, important and very useful. Coupled with Horrock's theorem above 3, one can easily complete the proof of Serre's problem. First, let me start with the notion of *extended modules.* If M is a module over $R[T]$, we say it is extended, if there exists an R-module N such that $M = N \otimes_R R[T]$. Of course, then $N \cong M/TM$.

Theorem 5 (Quillen). *Let M be a module over $R[T]$. If it is locally extended, that is, for every maximal ideal $\mathfrak{m} \subset R$, $M_\mathfrak{m}$ is extended from $R_\mathfrak{m}$, then M is extended.*

The proof of Quillen's theorem is elementary, but not easy. Quillen uses the variable T effectively in showing that if $u, v \in R$ and M_u, M_v are extended from $R_u[T]$ (resp. $R_v[T]$), then M_{u+v} is extended from $R_{u+v}[T]$. This is the main step of the proof.

4. Some applications

One of the main direct applications of Serre's problem was also suggested by Serre [Ser58], which nowadays is known as *Serre's Construction.*

If $X \subset \mathbb{A}^n$ is an affine variety, it is important to understand the ideal $I(X) \subset k[x_1, \ldots, x_n]$ of all polynomials vanishing on X. One of the crudest information one can have is the smallest number of generators of $I(X)$, usually denoted by $\mu(I(X))$. That any ideal of a polynomial ring has finitely many generators, is the content of Hilbert Basis Theorem [Hil65, Lan02]. That is to say, polynomial rings over a field are Noetherian. So, the problem is to study $\mu(I(X))$, given information about X. Serre indicated a method by which this can be done in certain cases, if one knew that projective modules are free over polynomial rings. We will discuss this in a little more detail below. Another tantalising question in this connection, attributed to Kronecker, is whether any curve in $\mathbb{A}^n$ can be set-theoretically defined by $n-1$ equations (the least possible by Krull's principal ideal theorem). In this generality, the problem is still open and we will discuss this too in this section and indicate, how the freeness of projective modules play a role in settling this question at least in many important cases.

Let me briefly indicate the salient features of Serre's construction. If $X \subset \mathbb{A}^n$ is a smooth subvariety of dimension $n-2$ (usually referred to as codimension two subvarieties) and $I = I(X)$, we want to find $\mu(I(X))$. Assume that X is defined by r equations in a neighbourhood of X. We would like to decide whether X can be defined by r equations in whole of $\mathbb{A}^n$. Serre showed that the local data , by homological algebra considerations, imply there exists a projective module of rank r over $\mathbb{A}^n$, which maps onto $I(X)$. So, since we can say that such a projective module is free and thus we have $\mu(I(X)) = r$, generated by the r elements of $I(X)$ obtained as the images of any basis of this free module. So, we have broken up the problem via this construction into two problems. One, how many eqaution do we need to define X is some neighbourhood of X (getting closer to intrinsic data) and then, the freeness of projective modules over polynomial rings. In fact, B. Segre [Seg70] used this construction to *show* that projective modules over $k[x, y, z]$ are not free, though the proof was incomplete. Murthy and Towber later settled it, as I mentioned.

S. S. Abhyankar showed in [Abh71] that if $X \subset \mathbb{A}^3$ is a smooth curve, then $I(X)$ is generated by three elements, the best possible in general. Abhyankar, in fact, more or less wrote down three very special elements in $I(X)$ which generate it. This result was used effectively by Murthy and Towber in their result on Serre's problem.

The idea of Serre was to find some intrinsic properties of X, which will give some estimates on $\mu(I(X))$ using his construction. For exmaple, Serre's construction would imply that if projective modules are free over $k[x, y, z]$, then any smooth curve in $\mathbb{A}^3$ is defined by three equations. Since the latter was known by Abhyankar's theorem, Murthy–Towber used it to show that projective modules over $k[x, y, z]$ are indeed free. So, one may think that we have gained nothing. This, fortunately is not true. The reason is that Serre's construction yields much more. For example, it will also imply that if $X \subset \mathbb{A}^3$ is a smooth curve which has a nowhere vanishing 1-form (a purely intrinsic data), then X is actually defined by *two* equations, if projective modules were free. This is usually expressed as such curves are *complete intersections*. This does not follow from Abhyankar's theorem, unless one goes through Murthy–Towber theorem. At any rate, these two notions, Serre's construction and solution to Serre's problem, imply nice estimates on $\mu(I(X))$ for many $X \subset \mathbb{A}^n$, which are intrinsic on X and not dependent on the specific embedding of X in $\mathbb{A}^n$.

Now, let me say a few words about Kronecker's problem. For those of you, who are wondering about the difference between computing $\mu(I(X))$ for an $X \subset \mathbb{A}^n$ and Kronecker's problem, let me start with an easy example.

Example: Let $C \subset \mathbb{A}^3$ be the curve defined as the common zeroes of the 2×2 minors of the following matrix.

$$\begin{pmatrix} x & y & z \\ y & z & x^2 \end{pmatrix},$$

where x, y, z are the coordinate functions on $\mathbb{A}^3$. Then one easily checks that these 3 equations are minimal and thus $\mu(I(C)) = 3$. But, as a set, C can be defined by the following *two* equations.

$$\det \begin{pmatrix} x & y \\ y & z \end{pmatrix} \text{ and } \det \begin{pmatrix} x & y & z \\ y & z & x^2 \\ z & x^2 & 0 \end{pmatrix}$$

As the above example illustrates, $\mu(I)$ can be larger than the the number of equations needed to define a variety set-theoretically. It is, for example easy to construct smooth curves in $\mathbb{A}^3$ which need exactly three equations to generate its ideal (as I mentioned in connection with Abhyankar's theorem above. But, we will see now that they are always defined by two equations set-theoretically, validating Kronecker. More generally, smooth curves in $\mathbb{A}^n$ are defined by $n-1$ equations, set theoretically. This was proved by D. Ferrand for $n = 3$ [Szp79, page 75] and by the author for $n \geq 4$ [Kum78, Corollary 5, page 235]. A similar result is also known for arbitrary curves (singular) at least in positive characteristic by a result of Cowsik and Nori [CN78]. All of these results crucially depend on the freeness of projective modules. Whether any curve in n-space is set-theoretically defined by $n-1$ equations over a characteristic zero field remains unknown at present.

As I have said earlier, projective modules being locally free, the issues where they rear their heads are global. So, often, if we had local data on number of generators of an ideal (or more generally of a module) from which we want to deduce similar information globally, we end up with a projective module mapping onto the ideal and not free modules in general, as in Serre's construction. The only difference in what we are discussing now, is that we are longer in a situation where our variety is of codimension two, an essential condition for using Serre construction. Let us say, you have a non-singular curve $C \subset \mathbb{A}^n$. The non-singularity is a local data. It tells you in particular, that the curve can be defined by $n-1$ equations locally (local coordinates, akin to implicit function theorem). Is it possible to put this data together globally? As I have mentioned, in general the ideal of the curve may not be generated by $n-1$ equations globally, since that would imply that the curve has a nowhere vanishing 1-form, which may not hold. But, the curve can be defined by n equations globally! But, how do we prove a result like this? First, since C is defined by $n-1$ equations locally and $\dim C = 1$, one can put these together fairly easily to show that there exists a single open set (remember Serre construction?) $C \subset U \subset \mathbb{A}^n$, where C is defined by n equations. (For larger dimensional varieties, the same principle of adding the number of local generators to the dimension of the variety applies and we can find a suitable open set containing the variety, where the variety is defined by certain number of equations, depending only on the variety in question and the dimension of the affine space where it is embedded, not how it is embedded). Of course, if the open set we found, $U = \mathbb{A}^n$, we have nothing to do. In general, this cannot be ensured, by the crude nature by which we find the U, purely in an existential manner. So we need to cover $\mathbb{A}^n$ by U and another open set V suitably, to use our local information to get some information globally. One arranges this (not quite trivial) so that for a suitable open set V with $V \cap C = \emptyset$ and $U \cup V = \mathbb{A}^n$, the n functions which define C on U, satisfy an additional property. Since these functions define C in U, they have no common zeroes on $U \cap V$ and thus define a unimodular row on $U \cap V$. We would arrange U, V so that this unimodular row actually gives a *free module* over $U \cap V$. This information now can be used to do a 'patching' to say that there exists a projective module of rank n mapping onto $I(C)$ and if projective modules were free, we see that $I(C)$ is generated by n functions. This is a typical procedure of going from local data to global information and if projective modules were not free, these results will still be interesting, but that they are free give much more information on these ideals defining varieties in affine spaces.

Seshadri's work also has inspired various mathematicians and though he himself did not work on this field afterwards, the torch has been carried ably by several younger people (some of whom are no longer so young) in Tata Instituite and the rest of the world. The work of Seshadri is possibly a starting point for the developement of Algebraic K-theory, by H. Bass [Bas68], a subject which has wide ramifications in various branches of Mathematics today. We have already seen that Hilbert Syzygy theorem implies a statement about K_0. Similarly, K_1 is defined in terms of the group $\mathrm{GL}(R)$ and how far is it from the (normal) subgroup $\mathrm{E}(R)$ and we have seen a result of this form in Suslin's theorem stated above. K-theory, in addition to being a very active field, has also become the language of choice for mathematicians. For, example theorems like the Grothendieck-Riemann-Roch are stated in the language of K-theory.

I hope, I have imparted some flavour of the Mathematics that has been influenced by Seshadri's work and have inspired at least some of you to delve deeper into this active and fertile field.

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