Lecture 1 : Some finite sums

Sum notation: If $a_1, \dots, a_n$ are real #'s (more on these later), then $\sum_{k=1}^{n} a_k := a_1 + a_2 + \dots + a_n$. We have the basic rules

- $\sum_{k=1}^{n} 1 = 1 + \dots + 1 = n$
- $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$ for any $c \in \mathbb{R}$
- $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ for $\underbrace{\{b_1, \dots, b_n\}}_{\text{set consisting of these elements}} \subset \mathbb{R}$

is an element of

real numbers

is a subset of

Here are some (possibly familiar) sums which we shall use to introduce the concept of mathematical induction.

(1) $S_a(n) := \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1-a}$ $\quad (a \in \mathbb{R}$ but $\neq 1)$

$$
\begin{aligned}
(1-a) S_a(n) &= S_a(n) - a S_a(n) \\
&= 1 + a + \dots + a^n \\
&\quad - (a + \dots + a^n + a^{n+1}) \\
&= 1 - a^{n+1}.
\end{aligned}
$$
Now divide.

A consequence (to be used in lecture 2):

$$
(*)\begin{cases} S_{\frac{1}{4}}(n) + \frac{1}{3} \cdot \frac{1}{4^n} = \dfrac{1 - \frac{1}{4^{n+1}}}{\frac{3}{4}} + \dfrac{1}{3 \cdot 4^n} = \dfrac{4}{3} - \dfrac{1}{3 \cdot 4^n} + \dfrac{1}{3 \cdot 4^n} \\ = \dfrac{4}{3} \end{cases}
$$

(2) $P_1(n) := \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$

1st approach (Gauss at age 7): $1+2+3+\cdots+(n-2)+(n-1)+n$

if n even, $\frac{n}{2}$ pairs with sum $n+1$

(if n odd, have $\frac{n+1}{2}$ in the middle, trickier but argument still works)

2nd approach (less intuitive but generalizes better):

$$(k+1)^2 - k^2 = 2k+1$$

$$\overset{\text{sum}}{\Longrightarrow} \underbrace{\sum_{k=1}^{n}\left((k+1)^2-k^2\right)}_{\text{collapses to } (n+1)^2-1^2 = n^2+2n \text{ (why?)}} = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = 2P_1(n) + n$$

$$\Longrightarrow n^2 + \cancel{2}n = 2P_1(n) + \cancel{n} \Longrightarrow P_1(n) = \frac{n^2+n}{2}$$

(3) $P_2(n) := \sum_{k=1}^{n} k^2 = \ ?$ Try same approach:

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1$$

$$\overset{\sum_{k=1}^{n}}{\Longrightarrow} (n+1)^3 - 1^3 = 3P_2(n) + 3P_1(n) + n$$

$$\Longrightarrow n^3 + 3n^2 + 3n = 3P_2(n) + \frac{3}{2}n^2 + \frac{5}{2}n$$

$$\Longrightarrow n^3 + \frac{3}{2}n^2 + \frac{1}{2}n = 3P_2(n)$$

$$\Longrightarrow P_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+1)}{6}$$

[Note: $P_r(n) = \sum_{k=1}^{n} k^r$ are the rth power sums.

There are formulas for these for every r, like

$$P_{10}(n) = \frac{n^{11}}{11} + \frac{n^{10}}{2} + \frac{5n^9}{6} - n^7 + n^5 - \frac{n^3}{2} + \frac{5n}{66};$$

in fact, there is a single formula for all of them which involves binomial coefficients & Bernoulli numbers.

See http://www.math.wustl.edu/~matkerr/ASMI.pdf]

Mathematical induction à la Apostol:

- A set S of real #'s is an <u>inductive set</u> if
  (a) $1 \in S$ & (b) for every $x \in S$, $x+1 \in S$
- A real # is called a <u>positive integer</u> if it belongs to every inductive set.
  Clearly the set $\mathbb{P}$ of these
  (a) contains 1, since 1 belongs to every inductive set
  (b) contains $1+1$, $1+1+1$, etc.
  $\Rightarrow$ $\mathbb{P}$ is an inductive set (the smallest one!)
- <u>Principle of Mathematical Induction</u>: Let S be a set of positive integers ($S \subseteq \mathbb{P}$). Suppose $\begin{cases} (a) & 1 \in S \\ (b) & k \in S \Rightarrow k+1 \in S. \end{cases}$
  Then every positive integer is in S ($\mathbb{P} \subseteq S$; and so $\mathbb{P} = S$).
  <u>Proof</u>: (a) & (b) $\Rightarrow$ S is an inductive set $\Rightarrow$ $\mathbb{P} \subseteq S$ by definition. $\square$
- <u>Inductive proofs</u>: Let $A(n)$ be an assertion about the integer n.
  Suppose we know (a) $A(1)$ holds ["base case"]
  & (b) $A(k)$ holds $\Rightarrow A(k+1)$ holds ("inductive step") for each integer $k \geq 1$
  can also do $A(k-1) \Rightarrow A(k)$
  Then $A(n)$ holds for every $n \geq 1$.
  [Why? By applying P. of M.I. to the set S of positive integers n for which $A(n)$ holds.]

Let's reprove the above sum formulas using induction:

(I) $A(n)$ is the assertion that $S_a(n) = \dfrac{1-a^{n+1}}{1-a}$.

<u>$A(1)$</u>: $1 + a = \dfrac{1-a^2}{1-a}$ ✓

<u>$A(n-1) \Rightarrow A(n)$</u>: $\underbrace{S_a(n-1)}_{1+a+\cdots+a^{n-1}} = \dfrac{1-a^n}{1-a}$ $\overset{\text{add } a^n}{\Longrightarrow}$ $S_a(n-1) + a^n = \dfrac{1-a^n}{1-a} + a^n$

$\Rightarrow S_a(n) = \dfrac{1-a^n + a^n(1-a)}{1-a} = \dfrac{1-a^n+a^n-a^{n+1}}{1-a} = \dfrac{1-a^{n+1}}{1-a}$. ✓

(2) $A(n)$ is the assertion that $P_1(n) = \frac{n^2}{2} + \frac{n}{2}$.

$\underline{A(1)}$: $1 = \frac{1^2}{2} + \frac{1}{2}$ ✓

$\underline{A(n-1) \Rightarrow A(n)}$: $P_1(n-1) \overset{A(n-1)}{=} \frac{(n-1)^2}{2} + \frac{n-1}{2} = \frac{n^2}{2} - n + \cancel{\frac{1}{2}} + \frac{n}{2} - \cancel{\frac{1}{2}}$

$$= \frac{n^2}{2} - \frac{n}{2}$$

$\xrightarrow[\text{add } n]{} P_1(n) = P_1(n-1) + n = \frac{n^2}{2} - \frac{n}{2} + n = \frac{n^2}{2} - \frac{n}{2}$ ✓

(3) $A(n)$ is the assertion that $P_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.

$\underline{A(1)}$: $1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6}$ ✓

$\underline{A(n-1) \Rightarrow A(n)}$: $P_2(n-1) \overset{A(n-1)}{=} \frac{(n-1)^3}{3} + \frac{(n-1)^2}{2} + \frac{n-1}{6}$

$$= \frac{n^3}{3} - n^2 + n - \cancel{\frac{1}{3}} + \frac{n^2}{2} - n + \cancel{\frac{1}{2}} + \frac{n}{6} - \cancel{\frac{1}{6}}$$

$$= \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

$\xRightarrow[\text{add } n^2]{} P_2(n) = P_2(n-1) + n^2 = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} + n^2$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$ ✓

(4) <u>An important inequality</u>: $\boxed{P_2(n-1) < \frac{n^3}{3} < P_2(n)}$ for all $n \geq 1$.

(**)

- This can be derived from (3):

$$P_2(n) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} > \frac{n^3}{3}$$

$$P_2(n-1) = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} = \frac{n^3}{3} - \underbrace{\frac{n}{6}(3n-1)}_{\substack{\text{positive} \\ (\text{if } n \geq 1)}} < \frac{n^3}{3}.$$

This was easy, but we needed to know a formula for $P_2(n)$. What if we <u>guessed</u> that (**) should be true and wanted to prove it <u>without</u> a formula for $P_2(n)$?

- Proof of ④ (call it $A(n)$) by induction:

$\underline{A(1)}$: $\quad 0 < \frac{1^3}{3} < 1$

$\underline{A(n) \Rightarrow A(n+1)}$: Given: $\underbrace{P_2(n-1) < \frac{n^3}{3}}_{(a)}$ and $\underbrace{\frac{n^3}{3} < P_2(n)}_{(b)}$.

Adding n^2 to (a) gives $P_2(n) < \frac{n^3}{3} + n^2 < \frac{n^3}{3} + n^2 + n + \frac{1}{3} = \frac{(n+1)^3}{3}$

Adding $(n+1)^2$ to (b) gives $P_2(n+1) > \frac{n^3}{3} + (n+1)^2 = \frac{n^3}{3} + n^2 + 2n + 1$

$$> \frac{n^3}{3} + n^2 + n + \frac{1}{3} = \frac{(n+1)^3}{3}.$$

<u>How not to use induction:</u> (can you see the problem?)

<u>Theorem:</u> All black cats have green eyes.

Since there is at least one black cat w/ green eyes, this will follow at once from the

<u>Lemma:</u> In any group of n black cats, at least one of which has green eyes, all n of them do.

<u>Proof:</u> Let $A(n)$ be the assertion. Clearly $A(1)$ holds.

We illustrate the inductive step by the case $A(3) \Rightarrow A(4)$:

suppose we have c_1, c_2, c_3, c_4 black cats, and c_1 has green eyes. Since $A(3)$ is assumed, we may apply it to $\{c_1, c_2, c_3\}$ ($\Rightarrow$ c_2 & c_3 have green eyes) & to $\{c_1, c_2, c_4\}$ ($\Rightarrow$ c_2 & c_4 have green eyes).

So $c_2, c_3,$ & c_4 all have green eyes, proving $A(4)$. $\square$