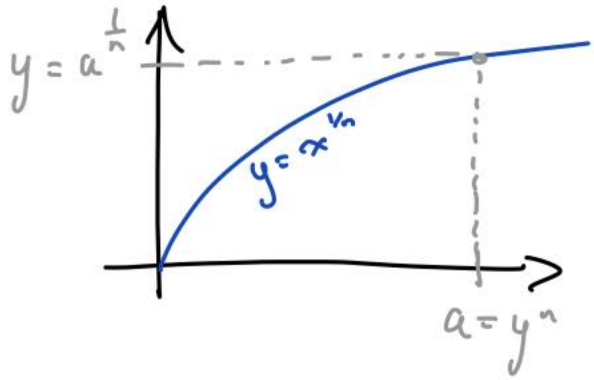
Lecture 10: Area & Volume integrals

Let's begin by computing a couple more integrals:

(A) Fractional powers



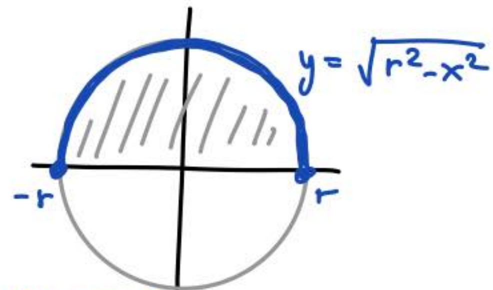
$$\Downarrow$$

$$a^{1+\frac{1}{n}} = \int_0^{a^{1/n}} y^n \, dy + \int_0^a x^{1/n} \, dx$$

$$\Downarrow$$

$$\int_0^a x^{1/n} \, dx = a^{1+\frac{1}{n}} - \frac{(a^{1/n})^{n+1}}{n+1} = \frac{a^{\frac{1}{n}+1}}{\frac{1}{n}+1}.$$

(B) Semicircles



Define $\pi := a\left(\substack{\text{unit} \\ \text{circle}}\right)$.

On $[-1, 1]$, with $f(x) = \sqrt{1-x^2}$,

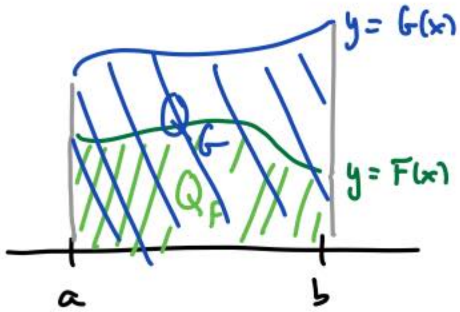
$$\frac{\pi}{2} = a(Q_f) = \int_{-1}^{1} \sqrt{1-x^2} \, dx \overset{\text{dilation property}}{=} \frac{1}{r} \int_{-r}^{r} \sqrt{1-\left(\frac{x}{r}\right)^2} \, dx = \frac{1}{r^2} \int_{-r}^{r} \sqrt{r^2-x^2} \, dx$$

$$\Longrightarrow \int_{-r}^{r} \sqrt{r^2-x^2} \, dx = \frac{\pi r^2}{2}.$$

Areas 1 : between two curves

Theorem: Let $f \le g$ be integrable on $[a,b]$, and set
$$\mathcal{S} := \{(x,y) \mid x \in [a,b],\ y \in [f(x), g(x)]\}.$$ Then
$$a(\mathcal{S}) = \int_a^b (g-f)(x)\,dx.$$

Proof: Write $F(x) = f(x) + C$, $G(x) = g(x) + C$ so $F, G \ge 0$, $G - F = g - f$, and $\mathcal{S}' := \{(x,y) \mid x \in [a,b],\ y \in [F(x), G(x)]\}$ is a translate of $\mathcal{S}$ ($\Rightarrow$ same area!). We have $\mathcal{S}' = Q_G \setminus Q_F^\circ$,

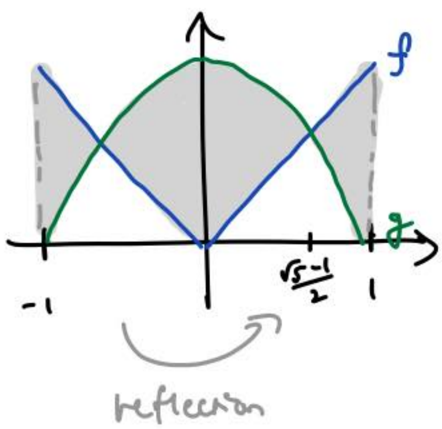


where $Q_G := \{(x,y) \mid x \in [a,b],\ y \in [0, G(x)]\}$
$\cup$
$Q_F^\circ := \{(x,y) \mid x \in [a,b],\ y \in [0, F(x))\}$

and $a(Q_F^\circ) = a(Q_F)$ since the same step integrals sandwich it (just use open-top rectangles for the lower sums). So $a(\mathcal{S}) = a(\mathcal{S}') = a(Q_G) - a(Q_F^\circ) =$
$$a(Q_G) - a(Q_F) = \int_a^b G(x)\,dx - \int_a^b F(x)\,dx = \int_a^b (G-F)(x)\,dx = \int_a^b (g-f)(x)\,dx.$$
$\square$

More generally, if we have $f \le g$ in part of $[a,b]$ & $g \le f$ in part of $[a,b]$, partition $[a,b]$ accordingly $\Longrightarrow$ $\boxed{\text{area between curves} = \int_a^b |f(x) - g(x)|\,dx.}$

Ex/ Find $A :=$ area between $f(x) = |x|$ & $g(x) = 1 - x^2$ on $[-1, 1]$.



By the symmetry shown, $A = 2\int_0^1 |f(x) - g(x)|\,dx$
$$= 2\int_0^{\frac{\sqrt{5}-1}{2}} (1 - x^2 - x)\,dx + 2\int_{\frac{\sqrt{5}-1}{2}}^1 (x - 1 + x^2)\,dx$$
$$= 2\left(x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_0^{\frac{\sqrt{5}-1}{2}} + 2\left(\frac{x^2}{2} - x + \frac{x^3}{3}\right)\Big|_{\frac{\sqrt{5}-1}{2}}^1 = \cdots$$
$$= \frac{5\sqrt{5} - 8}{3}.$$

Here $\frac{\sqrt{5}-1}{2}$ comes from solving $x = 1 - x^2$.

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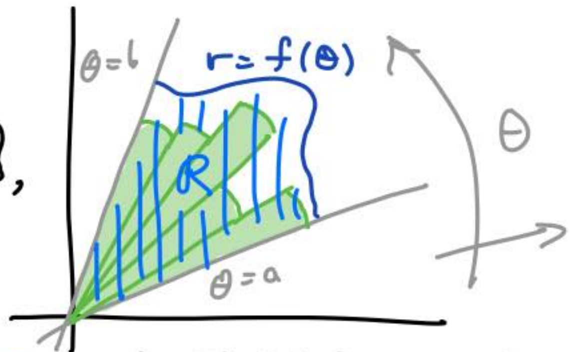
Areas 2 : polar coordinates



$(r,\theta) \leftrightarrow (x,y)$ via $x = r\cos\theta$, $y = r\sin\theta$

Let $R_f := \{(r\cos\theta, r\sin\theta) \mid \theta \in [a,b],\ r \in [0, f(\theta)]\}$, where $0 \le b-a \le 2\pi$ and $f \ge 0$ on $[a,b]$.

Theorem: If f^2 is integrable on $[a,b]$, then R is measurable and $\boxed{a(R) = \frac{1}{2}\int_a^b f^2(\theta)\,d\theta.}$

Proof: Recall that an angular sector with $\begin{cases}\text{angle } \theta \\ \text{radius } r\end{cases}$ had area $a(\sigma_{r,\theta}) = \frac{\theta}{2} r^2$ by definition. (We took $r=1$ before.) Given any step function $s \ge 0$ on $[a,b]$ with partition $P = \{\theta_0, \theta_1, \dots, \theta_N\}$ and values s_k on (θ_{k-1}, θ_k),

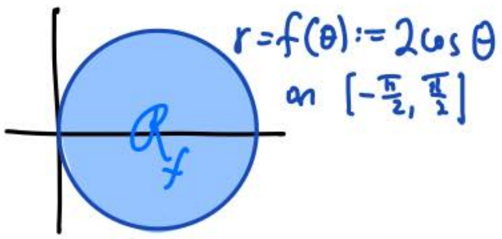
$$a(R_s) = \sum_{k=1}^N a(\sigma_{s_k, \theta_k - \theta_{k-1}}) = \frac{1}{2}\sum_{k=1}^N (\theta_k - \theta_{k-1}) s_k^2 = \frac{1}{2}\int_a^b s^2(\theta)\,d\theta.$$

Now for all step functions s & t with $0 \le s \le f \le t$ on $[a,b]$,

$$\frac{1}{2}\int_a^b s^2(\theta)\,d\theta = a(R_s) \le a(R_f) \le a(R_t) = \frac{1}{2}\int_a^b t^2(\theta)\,d\theta$$

(since $R_s \subseteq R_f \subseteq R_t$). Hence for all step functions s & t with $0 \le s \le \frac{1}{2}f^2 \le t$, $\int_a^b s(\theta)\,d\theta \le a(R_f) \le \int_a^b t(\theta)\,d\theta$. Since $\frac{1}{2}f^2$ is integrable (by assumption), $I(\frac{1}{2}f^2)$ is the unique number satisfying this — i.e. $a(R_f) = I(\frac{1}{2}f^2) = \frac{1}{2}\int_a^b f^2(\theta)\,d\theta$. $\square$

Ex/



Why is this a circle?

$$\begin{aligned}(x-1, y) &= (f(\theta)\cos\theta - 1, f(\theta)\sin\theta) \\ &= (2\cos^2\theta - 1, 2\cos\theta\sin\theta) \\ &= (\cos(2\theta), \sin(2\theta)).\end{aligned}$$

$$\begin{aligned}a(R_f) &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (2\cos\theta)^2\,d\theta \\ \overset{\text{symmetry}}{}&= 4\int_0^{\pi/2} \cos^2\theta\,d\theta \\ &= 2\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta \\ &= \int_0^{\pi/2} 2\,d\theta + 2\int_0^{\pi/2} \cos(2\theta)\,d\theta \\ \overset{\text{dilation}}{}&= 2\cdot\frac{\pi}{2} + \underbrace{\int_0^{\pi} \cos(\theta)\,d\theta}_{\sin\theta\big|_0^{\pi} = 0} \\ &= \pi.\end{aligned}$$

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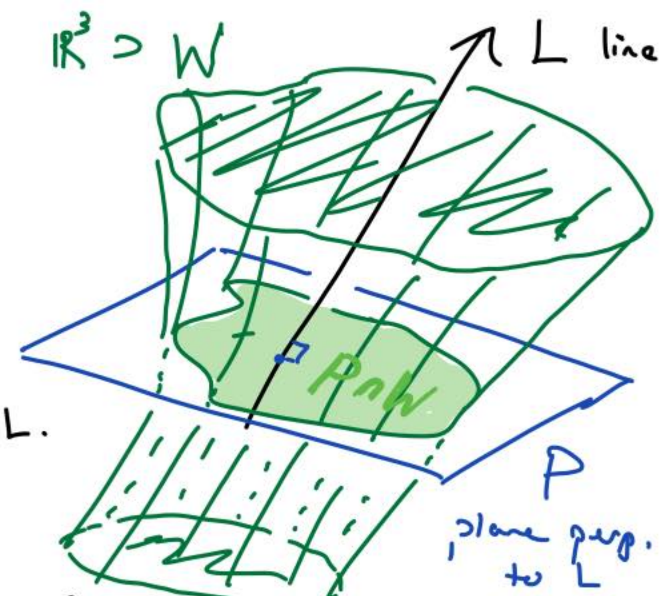
Areas 3: from area to volume



<u>Defn.</u>: A <u>Cavalieri solid</u> W is a subset of $\mathbb{R}^3 (= \mathbb{R}\times\mathbb{R}\times\mathbb{R})$ such that, for some line L, the cross sections $W \cap P$ are measurable for all planes P perpendicular to L.

<u>Axiomatic characterization of volume $v(W)$</u>:

$\exists$ set $\mathcal{A}$ of "measurable solids" (subsets of $\mathbb{R}^3$), containing all convex solids (e.g. boxes) and closed under union / intersection / difference, and a function $v : \mathcal{A} \to \mathbb{R}_{\geq 0}$ satisfying $v([0,a]\times[0,b]\times[0,c]) = abc$, & for every $W_1, W_2 \in \mathcal{A}$

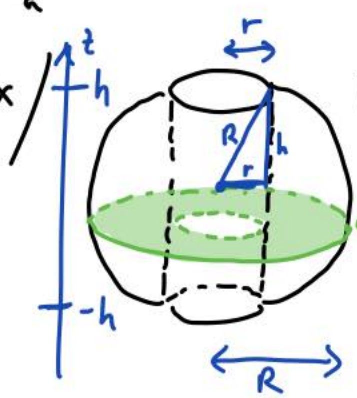
- $v(W_1 \cup W_2) = v(W_1) + v(W_2) - v(W_1 \cap W_2)$
- $W_1 \subseteq W_2 \Rightarrow v(W_2 \setminus W_1) = v(W_2) - v(W_1)$
- If W_1, W_2 are Cavalieri and $a(W_1 \cap P) \leq a(W_2 \cap P)$ $\forall P \perp L$, then $v(W_1) \leq v(W_2)$.

"Cavalieri Principle"

<u>Theorem</u>: Let $W \in \mathcal{A}$ be Cavalieri, with $A(u) := a(W \cap P_u)$ integrable on $[a,b]$ and zero outside this interval. Then $v(W) = \int_a^b A(u)\,du$.
(Here u is a coordinate on L; in the proof we may assume $L = z$-axis & $u = z$.)

<u>Proof</u>: Let s, t be any step functions with $s(z) \leq A(z) \leq t(z)$ on $[a,b]$. Construct solids $\mathcal{W}_s := \bigcup_{k=1}^{N} [z_{k-1}, z_k] \times R_k$ where R_k is a rectangle in (x,y) with area s_k (= value of s on (z_{k-1}, z_k)). Clearly $v(\mathcal{W}_s) = \sum_{k=1}^{N} (z_k - z_{k-1}) s_k$
$= \int_a^b s(z)\,dz$, similarly for $\mathcal{W}_t$. By Cavalieri we have now
$\int_a^b s\,dz = v(\mathcal{W}_s) \leq v(W) \leq v(\mathcal{W}_t) = \int_a^b t\,dz$; hence $v(W) = I(A)$. $\square$

Ex/



W = cored apple

area $= A(z) = \pi\left(\sqrt{R^2 - z^2}\right)^2 - \pi r^2 = \pi\left[R^2 - r^2 - z^2\right] = \pi\left[h^2 - z^2\right]$

By the Theorem, $v(W) = 2\int_0^h A(z)\,dz$

$= 2\pi \int_0^h (h^2 - z^2)\,dz = 2\pi\left[h^2 z - \frac{z^3}{3}\right]\Big|_0^h$

$= 2\pi\left[h^3 - \frac{1}{3}h^3\right] = \frac{4}{3}\pi h^3.$ //

Note: $h^2 = R^2 - r^2$.