Lecture 11: Indefinite integrals; Limits

Last applications of the definite integral

(All functions assumed integrable)

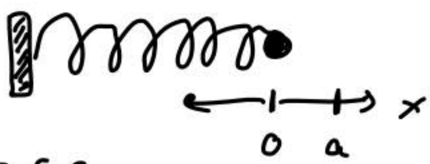
① Work

<u>Definition</u>: [physics] Energy expended by a force (= function of position) in moving an object from point "a" to point "b".

[math] $W = \int_a^b f(x)\,dx$.

[units: foot-lbs., Newton-meters, etc.]

Typical examples involve (i) Hooke's Law



(force to hold spring at $x=a$ is $f(a) = ca$, for some "spring constant" $c \in \mathbb{R}^+$. So work done in stretching spring from 0 to a is $ca^2/2$.)

(ii) Pulling a chain up onto a ledge
[See HW problem #5 on p.117]

② <u>Average value</u>/etc.

For $f: [a,b] \to \mathbb{R}$,

$$Avg(f) := \frac{1}{b-a}\int_a^b f(x)\,dx$$ is the <u>average value</u>.

But we can also weight the average: perhaps $P(x)$ is a probability distribution function, with $\int_{x_1}^{x_2} P(x)\,dx$ the probability x lies in $[x_1, x_2)$ and $\int_a^b P(x)\,dx = 1$ (= "100%."). Then the <u>weighted average</u> or <u>expected value</u> of f would be

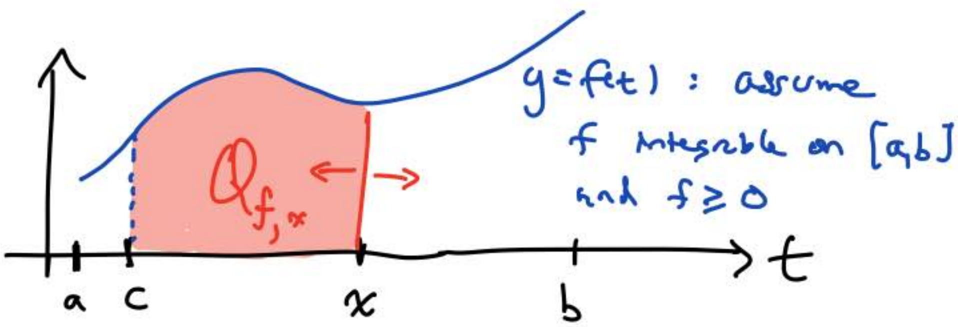
$$E(f) := \int_a^b f(x)P(x)\,dx.$$ ($Avg(f)$ is just the case $P(x) = \frac{1}{b-a}$.)

Finally, maybe the weighting function isn't a probability distribution, e.g. it might be a mass-density $\mu(x)$ with $\int_a^b \mu(x)\,dx$ different from 1. Then the weighted average is $\int_a^b f(x)\mu(x)\,dx \Big/ \int_a^b \mu(x)\,dx$. In the special case where $f(x) = x$ is the "position function", we get

$$\bar{x} := \int_a^b x\,\mu(x)\,dx \Big/ \int_a^b \mu(x)\,dx \quad = \text{ the center of mass.}$$

HW problem is #22 on p. 119.

Indefinite integrals



- In a picture:

  $F(x) := a(Q_{f,x})$

- As an integral: $F(x) := \int_c^x f(t)\,dt$.

If f isn't always ≥ 0, then the picture interpretation must add areas above the x-axis and subtract areas below the x-axis.

2 Key remarks:

(1) For $A, B \in [a,b]$,

$$\int_A^B f(x)\,dx = \int_A^a f(x)\,dx + \int_a^B f(x)\,dx = \int_a^B f(x)\,dx - \int_a^A f(x)\,dx$$

$$= F(B) - F(A) =: F(x)\Big|_A^B .$$

(2) There are as many choices of indefinite integral $F(x)$ of f (done this way) as there are choices of c.

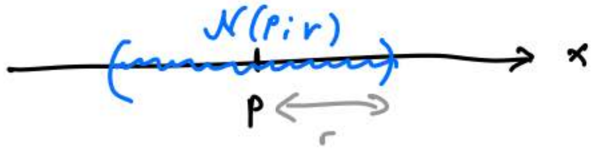
Limits (a first view)

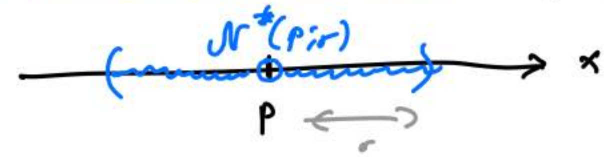
Next week we will be discussing continuity as we turn from integral toward differential calculus. To that end, we must first make precise the meaning of "$\lim_{x \to p} f(x) = L$".

<u>Notation</u>: $\mathcal{N}(p;r) := \{ x \in \mathbb{R} \mid |x-p| < r, \text{ i.e. } -r < x-p < r \}$

$[$resp. $\mathcal{N}^*(p;r) := \{ x \in \mathbb{R} \mid 0 < |x-p| < r \}]$

are called neighborhoods [resp. punctured neighborhoods] of $p \in \mathbb{R}$:





Sometimes we'll just write $\mathcal{N}(p)$ or $\mathcal{N}^*(p)$ if we don't want to specify the "radius" $r \in \mathbb{R}^+$. (Also, we can enumerate them: $\mathcal{N}_0, \mathcal{N}_1, \ldots$)

Now let $f : D \to \mathbb{R}$ be a function with domain $D \subset \mathbb{R}$. Assume that D contains some punctured neighborhood of $p \in \mathbb{R}$. (D <u>need not contain</u> p itself, just some $\mathcal{N}_0^*(p)$.) Then

$$\lim_{x \to p} f(x) = L \qquad \text{means that}$$

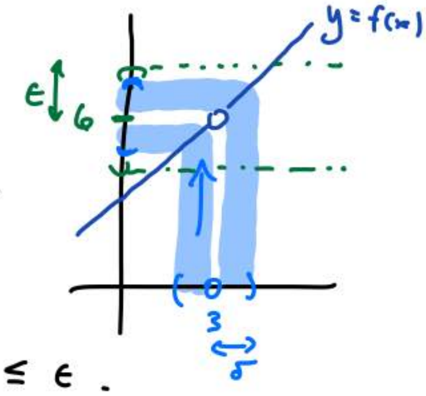
- for every neighborhood $\overbrace{\mathcal{N}(L)}^{\text{"no matter how small"}} \subset \mathbb{R}$ of L, there exists a punctured neighborhood $\underbrace{\mathcal{N}^*(p) \subset D}_{\text{"sufficiently small"}}$ such that

$$\underbrace{x \in \mathcal{N}^*(p) \implies f(x) \in \mathcal{N}(L)}_{\text{you can read this: } f(x) \in \mathcal{N}(L) \text{ whenever } x \in \mathcal{N}^*(p).}.$$

By making the radii of the two neighborhoods explicit, this becomes (i.e. is equivalent to):

- for every $\overset{\text{radius of } \mathcal{N}(L)}{\epsilon} > 0$, there exists a $\overset{\text{radius of } \mathcal{N}^*(p)}{\delta} > 0$ such that

$$0 < |x-p| < \delta \implies |f(x) - L| < \epsilon .$$



Ex/ $f(x) = \dfrac{x^2-9}{x-3}$, defined on $D = \mathbb{R} \setminus \{3\}$.

Since for $x \in D$, $f(x) = x+3$, taking $\delta \leq \epsilon$

gives $0 < |x-3| < \delta \implies |f(x) - 6| = |x-3| < \delta \leq \epsilon$.

So $\lim\limits_{x \to 3} f(x) = 6$. //

Non-Ex/ The following "do not exist": there is no number L that makes the statement $\lim\limits_{x \to p} f(x) = L$ true:

- $\lim\limits_{x \to 0} \dfrac{1}{x^2}$ (no matter how close you take x to 0, $f(x)$ takes values much bigger than any fixed L)

- $\lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right)$ (no matter how close you take x to 0, $f(x)$ still takes all values in $[-1, 1]$)

- $\lim\limits_{x \to 0} \dfrac{x}{|x|}$ (no matter how close you take x to 0, $f(x)$ still takes the set of values $\{-1, 1\}$; for $\epsilon < 1$ these can't both be within ϵ of any L)

Note that the problem is <u>NOT</u> that f isn't defined at $x=0$. We could actually define $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ c & x = 0 \end{cases}$ and that makes no difference for the nonexistence of the limit. Likewise,

$f(x) := \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 1 \text{ billion} & \text{if } x = 3 \end{cases}$ still has $\lim\limits_{x \to 3} f(x) = 6$.