Lecture 12 : Limits & continuity

Recall from last lecture that for a function f with domain D containing a punctured neighborhood of p, and $L \in \mathbb{R}$,

$$\lim_{x \to p} f(x) = L$$

means that

for each given $\epsilon > 0$, there exists some $\delta > 0$ such that

$$0 < |x - p| < \delta \implies |f(x) - L| < \epsilon .$$

$$\left(x \in (p-\delta, p+\delta) \setminus \{p\}\right) \qquad \left(f(x) \in (L-\epsilon, L+\epsilon)\right)$$

Basic examples : $f(x) = C$ (constant) — any δ works

$f(x) = x$ — $\delta = \epsilon$ works

Basic non-examples : did some of these last time. For instance,

Suppose $\lim_{x \to 0} \frac{1}{x^2} = L$, for some real # L. Then given (say) $\epsilon = 1$, there exists δ s.t. $x \in (0, \delta) \implies \frac{1}{x^2} \in (L-1, L+1) \implies \frac{1}{x^2} < L+1$
$\implies L+1 \in \mathbb{R}^+$ and $x > \frac{1}{\sqrt{L+1}}$, which is a contradiction b/c we took any $x \in (0, \delta)$!

How do we get more examples? By the

$\boxed{\text{LIMIT LAWS}}$ Suppose $\lim_{x \to p} f(x) = L_f$, $\lim_{x \to p} g(x) = L_g$ exist.

(A) $\lim_{x \to p} (f(x) \pm g(x)) = \lim_{x \to p} f(x) \pm \lim_{x \to p} g(x)$

(B)-(C) $\lim_{x \to p} f(x) g(x) = \lim_{x \to p} f(x) \lim_{x \to p} g(x)$, $\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$ provided this isn't 0.

(D) If $f \le H \le g$ for $x \in \mathcal{N}^*(p)$, and $L_f = L = L_g$, then $\lim_{x \to p} H(x) = L$.

<u>Proof of (A)</u>: Let $\epsilon > 0$ be given. There exist

- δ_f s.t. $0 < |x-p| < \delta_f \Rightarrow |f(x) - L_f| < \epsilon/2$
- δ_g s.t. $0 < |x-p| < \delta_g \Rightarrow |g(x) - L_g| < \epsilon/2$.

So if I take $\delta := \min\{\delta_f, \delta_g\}$, both of these hold, and

$$\begin{aligned}
|(f(x) \pm g(x)) - (L_f \pm L_g)| &= |(f(x) - L_f) \pm (g(x) - L_g)| \\
&\leq |f(x) - L_f| + |g(x) - L_g| \quad \text{(triangle inequality)} \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

$\square$

<u>Proof of (B)</u>: Let $\epsilon > 0$ be given, and put $C_\epsilon := \min\left\{\frac{\epsilon}{1+|L_f|+|L_g|}, 1\right\}$.

As above, there exists a "common" δ s.t. $0 < |x-p| < \delta \Rightarrow$

$$|f(x) - L_f|,\ |g(x) - L_g| < C_\epsilon .$$

So then
$$\begin{aligned}
|f(x)g(x) - L_f L_g| &= |(f-L_f)g + (g-L_g)L_f| \\
&= |(f-L_f)(g-L_g) + L_g(f-L_f) + L_f(g-L_g)| \\
&\leq |f-L_f||g-L_g| + |L_g||f-L_f| + |L_f||g-L_g| \\
&< C_\epsilon^2 + |L_g| C_\epsilon + |L_f| C_\epsilon \\
&= C_\epsilon (C_\epsilon + |L_g| + |L_f|) \\
&\leq C_\epsilon (1 + |L_g| + |L_f|) \\
&\leq \frac{\epsilon}{1+|L_g|+|L_f|}(1+|L_g|+|L_f|) = \epsilon. \quad \square
\end{aligned}$$

How did I think of this? Wrote it out without knowing how to choose C_ϵ, then realized what it needed to be when reached this line. See Apostol for a different proof.

<u>Application</u>: Define $f(x)$ to be <u>continuous</u> at a point $p \in \mathbb{R}$ if (i) $\lim_{x\to p} f(x)$ exists, (ii) $f(p)$ exists ($p \in D$), and (iii) $\lim_{x \to p} f(x) = f(p)$.

$f(x)$ is "continuous" if it is continuous at all points in its domain D.

(A)-(B) say that sums & products of continuous functions are continuous.

Since $f(x) = C$ & $f(x) = x$ are continuous, from this we get that all polynomial functions are continuous!

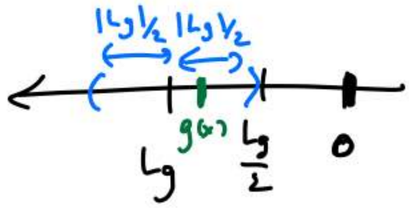
Proof of (C): If we show $\lim_{x\to p} \frac{1}{g(x)} = \frac{1}{L_g}$, we can just apply (B).

Given $\epsilon > 0$, let $C = \min\left\{\frac{\epsilon L_g^2}{2}, \frac{|L_g|}{2}\right\}$ and pick $\delta > 0$ s.t.

$$0 < |x-p| < \delta \implies |g(x) - L_g| < C.$$

Then

$$|g(x) - L_g| < \frac{|L_g|}{2} \implies |g(x)| \underset{(*)}{\geq} \frac{|L_g|}{2},$$



and

$$\left|\frac{1}{g(x)} - \frac{1}{L_g}\right| = \frac{|L_g - g|}{|g||L_g|} < \frac{C}{|g||L_g|} \leq \frac{\epsilon L_g^2/2}{|g|\,|L_g|} = \frac{\epsilon|L_g|/2}{|g|} \underset{(*)}{\leq} \frac{\epsilon|L_g|/2}{|L_g|/2} = \epsilon. \quad \square$$

Application of (C): Quotients $f(x)/g(x)$ of continuous functions are continuous on the complement of the set where $g(x) = 0$. Since polynomials are continuous, so therefore are rational functions $\frac{P(x)}{Q(x)}$ ← polynomials where $Q(x) \neq 0$.

Proof of (D): Given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $0 < |x-p| < \delta \implies$ $|f(x) - L|, |g(x) - L| < \epsilon/3$. Since $f \leq H \leq g \implies 0 \leq g - H \leq g - f$, this yields

$$|H(x) - L| = |(H-g) + (g-L)| \leq |H-g| + |g-L|$$
$$\leq |g-f| + |g-L| \leq |g-L| + |L-f| + |g-L|$$
$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \quad \square$$

Application of (D): $\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$ for $x \in N^*(0; \pi/2)$.

We'll show in a moment that $\cos(x)$ is continuous; so $\lim_{x\to 0} \cos(x) = 1$. From (C) it follows that $\lim_{x\to 0} \frac{1}{\cos(x)} = 1$. So by (D), $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1.}$

The following result gives (together with (C)) the continuity of all fractional powers and trigonometric functions where they are defined, since they all arise as integrals of bounded functions.

Theorem: If $f(x)$ is integrable on $[a,b]$, $F(x) := \int_a^x f(t)\,dt$ is continuous on $[a,b]$.

Proof: Since f is integrable, it is bounded. So $\exists\, M \in \mathbb{R}^+$ with $M \geq |f(t)|$ on $[a,b]$. Thinking of $-M \leq f \leq M$ as lower & upper step functions, we get

$$M\,|x-p| \geq \left| \int_p^x f(t)\,dt \right| = \left| F(x) - F(p) \right| .$$

Given $\epsilon > 0$, we take $\delta := \frac{\epsilon}{M}$: this yields

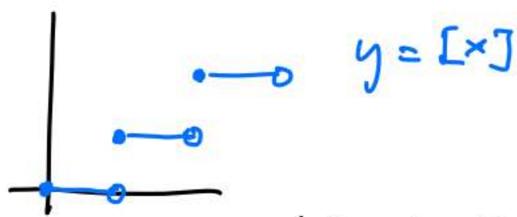
$$0 < |x-p| < \delta \implies |F(x) - F(p)| \leq M|x-p| < M\delta = M\frac{\epsilon}{M} = \epsilon .$$

$\square$

Remarks: We can also talk about left or right limits & left or right continuity : simply replace everywhere

- for "left" : $\lim\limits_{x\to p}$ by $\lim\limits_{x\to p^-}$, $0<|x-p|<\delta$ by $x \in (p-\delta, p)$
- for "right" : $\lim\limits_{x\to p}$ by $\lim\limits_{x\to p^+}$, $0<|x-p|<\delta$ by $x \in (p, p+\delta)$.

For instance, the greatest integer function has $\lim\limits_{x\to 1^+} [x] = 1$, $\lim\limits_{x\to 1^-} [x] = 0$.



Since $[1] = 1$, $[x]$ is right-continuous but not left-continuous at 1. In general: "continuity" is the same as "left-continuity" + "right-continuity"; and existence of $\lim\limits_{x\to p} f(x)$ is the same as existence and equality of the left & right limits.

Reason for this is simply that $N^*(p;\delta) = (p-\delta, p) \cup (p, p+\delta)$.

used to define limit

used to define left limit

used to define right limit.