Lecture 13: More on continuous functions

Ex / How do we extend $F(x) := \frac{\sin(2x)}{x}$ (on $\mathbb{R}\setminus\{0\}$) to a continuous function on $\mathbb{R}$?

Ideally, we would write $\frac{\sin(2x)}{x} = \frac{\sin(2x)}{2x} \cdot 2$ and then argue that $\lim_{x\to 0} \frac{\sin(2x)}{2x}$ is the "same" as $\lim_{y\to 0} \frac{\sin(y)}{y}$. While this is intuitively clear, and you could write out a limit argument, it seems better to prove a more general result which Apostol seems to overlook.

<u>Definition</u>: Let f be a function with domain D, and g be a function with domain E. The <u>composition</u> $f \circ g$ is defined by $(f\circ g)(x) := f(g(x))$, and has domain given by $\mathrm{Dom}(f\circ g) = \{x \in E \mid g(x) \in D\}$.

<u>Theorem</u>: (Limit law for compositions)

(i) If $\lim_{x\to p} g(x)$ is defined (we need not have $p \in E$) and f is continuous at $\lim_{x\to p} g(x)$, then $\lim_{x\to p}(f\circ g)(x) = f(\lim_{x\to p} g(x))$.

(ii) If g is continuous at p, and $\lim_{y\to g(p)} f(y)$ exists, then $\lim_{x\to p}(f\circ g)(x) = \lim_{y\to g(p)} f(y)$ provided that $g(x) \neq g(p)$ for $x \in N^*(p)$ (i.e. x near but not equal to p).

<u>Proof</u>: (i) Write $L_g := \lim_{x\to p} g(x)$. By hypothesis, $\lim_{y\to L_g} f(x) = f(L_g)$.

So given $\epsilon > 0$, there exists $\tau > 0$ s.t.

(*) $|y - L_g| < \tau \implies |f(y) - f(L_g)| < \epsilon$.

Moreover, thinking of this τ as "the ϵ" for $g(x)$, $\exists\, \delta > 0$ s.t.

$0 < |x-p| < \delta \implies |g(x) - L_g| < \tau$.

But then taking $y = g(x)$ in $(*)$, we get $|f(g(x)) - f(L_g)| < \epsilon$ as desired. So $\lim_{x \to p} f(g(x)) = f(L_g)$.

(ii) Write $L := \lim_{y \to g(p)} f(y)$, and let $\epsilon > 0$ be given; by defn. of limit $\exists\, \tau > 0$ s.t.

$$(**) \qquad 0 < |y - g(p)| < \tau \implies |f(y) - L| < \epsilon.$$

By continuity of g together with the additional hypothesis, $\exists\, \delta > 0$ s.t.

$$0 < |x - p| < \delta \implies \underline{\underline{0 <}} |g(x) - g(p)| < \tau.$$

Taking $y = g(x)$ in $(**)$, we get $|f(g(x)) - L| < \epsilon$, done. $\square$

Now write $F(x) = f(g(x))$, where $g(x) = 2x$ and $f(y) = 2\frac{\sin(y)}{y}$.

Applying (ii), $\lim_{x \to 0} F(x) = \lim_{y \to g(0) = 0} f(y) = \lim_{y \to 0} 2 \cdot \lim_{y \to 0} \frac{\sin(y)}{y} = 2 \cdot 1 = 2.$ //

<u>Corollary (of the Theorem)</u>: If g is continuous at p and f is continuous at $g(p)$, then $f \circ g$ is continuous at p.

(This is just (i) in the special case where g is continuous at p.)

Ex/ Let $f(x) = \sin(x)$, $g(x) = \sqrt{x}$ with domain $\mathbb{R}_{\geq 0}$.

$(f \circ g)(x) = \sin(\sqrt{x})$ is continuous w/ domain $\mathbb{R}_{\geq 0}$.

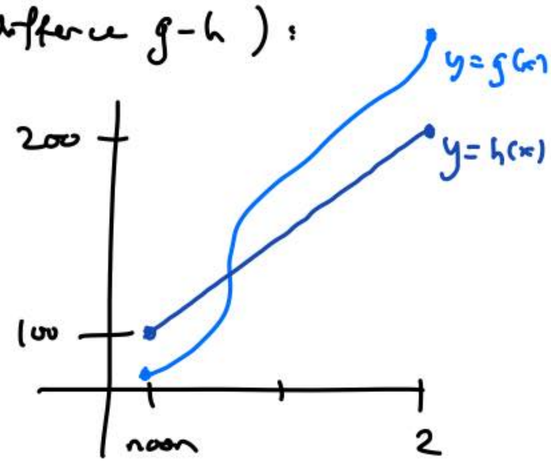
$(g \circ f)(x) = \sqrt{\sin(x)}$ is continuous w/ domain $\bigcup_{k \in \mathbb{Z}} [2k\pi, (2k+1)\pi]$. (Why?)

You should convince yourself they aren't the same!

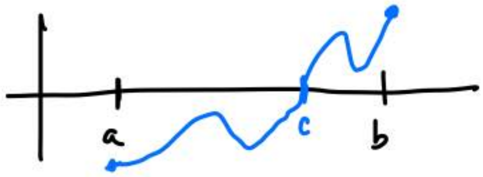
Composition is not a commutative operation: $f \circ g \neq g \circ f$ in most cases. But it is associative in the sense that $(f \circ g) \circ h = f \circ (g \circ h)$. //

Intermediate Value Theorem

Suppose you're traveling on the highway, starting before mile marker 100 at noon and ending up well past mile marker 200 at 2 PM. There is a cop car starting at mile marker 100 at noon and driving a constant 50 MPH until 2 PM. The fact that you are definitely going to get radared is a consequence of the following (applied to the difference $g-h$):



Bolzano's Theorem: If f is continuous on $[a,b]$, and $f(a)$ & $f(b)$ have opposite signs, then $\exists\, c \in (a,b)$ s.t. $f(c)=0$.



Lemma: If f is continuous at c and $f(c) \neq 0$, then $\exists\, \delta > 0$ s.t. $|x-c| < \delta \Rightarrow f(x)$ has the same sign as $f(c)$.

Proof: We may assume $f(c) > 0$ (Why?). Pick $\epsilon = \frac{f(c)}{2}$; then (by the continuity assumption) $\exists\, \delta > 0$ s.t. $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

But then $f(x) \in (f(c)-\epsilon, f(c)+\epsilon) \Rightarrow f(x) > f(c) - \epsilon = \frac{f(c)}{2} > 0$. $\square$

Proof of Bolzano: Assume $f(a) < 0$, $f(b) > 0$. Let $S := \{x \in [a,b] \mid f(x) \leq 0\}$; it is nonempty and bounded above, so $c := \sup S$ exists.

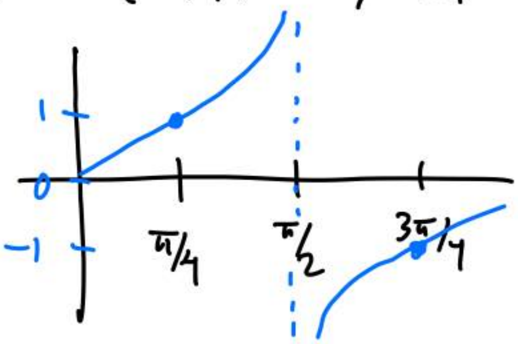
- Suppose $f(c) > 0$; then by the Lemma, $\exists\, \delta > 0$ s.t. $f > 0$ on $(c-\delta, c+\delta) \cap [a,b]$, while $f > 0$ on $(c, b]$. So $f > 0$ on $(c-\delta, b]$, and $c-\delta$ is an upper bound for S, contradicting the definition of c as the <u>least</u> upper bound.
- Suppose $f(c) < 0$. Then $f < 0$ on some $(c-\delta, c+\delta)$ (by the Lemma), but then (say) $c + \delta/2 \in S$ contradicting fact that c is an UB for S.
- Hence $f(c) = 0$.

$\square$

Intermediate Value Theorem: If f is continuous on $[a,b]$, then f assumes every value y_0 between $f(a)$ & $f(b)$ somewhere in (a,b).

Proof: Apply Bolzano to $f - y_0$. $\square$

Remark: Continuity on $[a,b]$ means in particular that f is right continuous at a & left-continuous at b. Without continuity, Bolzano & IVT fail! For example, let $f(x) := \tan(x)$: we have $f(\pi/4) = 1$ and $f(3\pi/4) = -1$, but nowhere in $(\pi/4, 3\pi/4)$ does $f(x) = 0$:



An application: Inverse functions

strictly increasing, or strictly decreasing

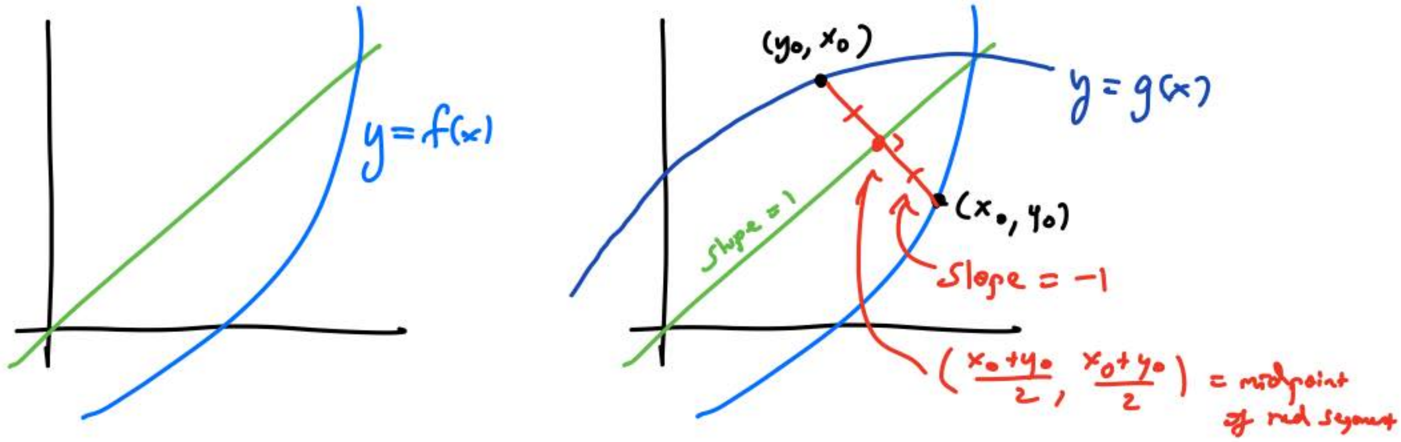
Proposition: Let f be a continuous, strictly monotonic function on $[a,b]$. Then there is a continuous, strictly monotonic function g on $\underbrace{[f(a), f(b)] \text{ (or } [f(b), f(a)])}_{\text{call this interval } I}$ such that $(g \circ f)(x) = x$.

Proof: Given any $y_0 \in I$, by the IVT there is some $x_0 \in [a,b]$ at which $f(x_0) = y_0$. By strict monotonicity, this x_0 is unique. So we can define $g(y_0) := x_0$. The argument that g is strictly monotonic is routine. For continuity at y_0, let $\epsilon > 0$ be given (and assume f increasing). Pick $\delta := \min\{y_0 - f(x_0 - \epsilon), f(x_0+\epsilon) - y_0\}$.

Then $y \in (y_0 - \delta, y_0 + \delta) \Rightarrow y \in (f(x_0 - \epsilon), f(x_0 + \epsilon)) \Rightarrow$

$g(y) \in (x_0 - \epsilon, x_0 + \epsilon) = (g(y_0) - \epsilon, g(y_0) + \epsilon)$. $\square$

This g, called the inverse of f, has graph Γ_g given by the reflection of Γ_f through the diagonal $y=x$ line:



Why is this? Notice that $y=g(x)$ is the same as (apply f to both sides & flip around) $x=f(y)$. That is,

$\Gamma_g = \{ (x,y) \mid (y,x) \in \Gamma_f \}$. The figure explains why swapping x & y gives the reflection.

Remark: We will sometimes write $g = f^{-1}$. (But this can also be a confusing notation, since f^{-1} can also mean $\frac{1}{f}$.) For instance, if $f(x) = x^n$, then $f^{-1}(x) = x^{1/n}$.

when I mean this, I will write it (and not "f^{-1}").

Ex/ Find f^{-1} if $f(x) = 2x+5$.

Think: $y=f(x)$, $x=f^{-1}(y)$. So I need to solve $y=2x+5$ for x:

$$y-5 = 2x$$
$$\tfrac{1}{2}(y-5) = x$$

$$\Rightarrow f^{-1}(y) = \tfrac{1}{2}(y-5).$$

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