Lecture 14: Mean and extreme values

Observation 1: The mean (= average) value of a function on a closed interval is between its extreme (= maximum & minimum) values.

(e.g.) During all of 2018, the max/min temperatures in St. Louis were $-6°F$ and $100°F$, and the average temp. was $57.3°F$.

More precisely, we have the

Theorem 1: For f integrable on $[a,b]$,

$$(*)\ \overbrace{\inf\{f(x) \mid x\in[a,b]\}}^{\text{"}\inf(f)\text{"}} \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \overbrace{\sup\{f(x)\mid x\in[a,b]\}}^{\text{"}\sup(f)\text{"}}.$$

This remains true with the middle term replaced by the weighted average
$$\frac{\int_a^b f(x)\,g(x)\,dx}{\int_a^b g(x)\,dx}$$
for any integrable $g \ge 0$ with integral $\ne 0$ such that fg is also integrable.

(Of course, $(*)$ is the case $g(x) = 1$.)

(e.g.) The center of mass of a rod $\underset{A}{[}\rule{3em}{0.4pt}\underset{B}{]}$ with mass density function $g(x)$, given by $\int_A^B x\,g(x)\,dx \Big/ \int_A^B g(x)\,dx$, is always between A and B (here $f(x) = x$).

(e.g.) Let $[a,b] = [0,1]$, $f(x) = \frac{1}{\sqrt{1+x}}$, $g(x) = x^9$. The inf & sup in Theorem 1 are $1/\sqrt{2}$ and 1, while $\int_0^1 g(x)\,dx = \frac{1}{10}$.

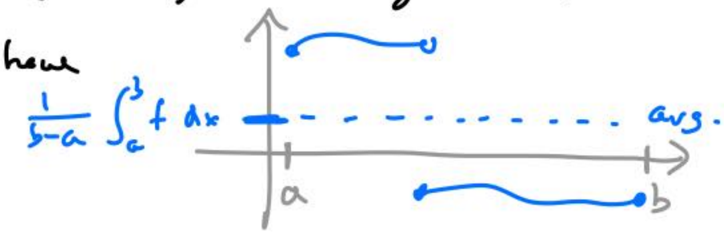
So Theorem $\Rightarrow$ $\frac{1}{10\sqrt{2}} \le \int_0^1 \frac{x^9}{\sqrt{1+x}}\,dx \le \frac{1}{10}$.

Proof of "Theorem": $m \le f(x) \le M$ on $[a,b]$ $\Rightarrow$ $m\,g(x) \le f(x)g(x) \le M\,g(x)$

$\Rightarrow$ $m\int_a^b g\,dx \le \int_a^b fg\,dx \le M\int_a^b g\,dx$. Divide by $\int_a^b g\,dx$. $\square$

Observation 2: If f is continuous, then it should actually attain this average value (or weighted average value) somewhere.

(In other words, we can't have



)

This "contains" two theorems:

Theorem 2: If f is continuous, it is integrable.

Theorem 3: If f & g are continuous on $[a,b]$, with $g \geq 0$ & $\int_a^b g\,dx > 0$, then there exists a $c \in [a,b]$ s.t.

Mean Value Thm. for Integrals.

$$f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}.$$

If $g = 1$, this reads $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

(e.g.) Suppose f is continuous and $\int_a^b f(x)\,dx = 0$. Then $\exists\ c \in [a,b]$ with $f(c) = 0$. (Take $g = 1$ in Thm. 3.)

Observation 3: Continuous fcns. on closed intervals attain their extreme values.

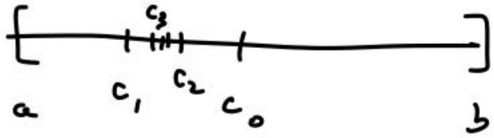
Theorem 4: If f is continuous on $[a,b]$, then there exist $d, e \in [a,b]$ s.t. $f(d) = \inf(f)$ & $f(e) = \sup(f)$.

Proof of Thm. 3 assuming 2 & 4: Since (by Th. 1) $\frac{\int_a^b fg\,dx}{\int_a^b g\,dx}$ is between $f(d)$ & $f(e)$, the statement follows from the Intermediate Value Theorem for continuous functions. $\square$

Proof of Theorem 4 assuming 2: Set $F(x) := \sup(f) - f(x)$ (≥ 0) and suppose $F > 0$ on $[a,b]$. Then $\frac{1}{F}$ is continuous (why?) and (by Thm. 2 integrable hence) bounded: i.e. $\exists\ B \in \mathbb{R}^+$ s.t. $\frac{1}{F} \leq B$
$\Rightarrow F \geq \frac{1}{B} \Rightarrow f \leq \sup(f) - \frac{1}{B}$ on $[a,b]$, contradicting $\sup(f)$'s minimality as an upper bound. So $F(d) = 0$ for some d. $\square$

<u>Proof of Theorem 2</u>: We need to show that (a) f is bounded and (b) $\underline{I}(f) = \overline{I}(f)$.

(a) Suppose otherwise. Then f is unbounded in $[a, c_0]$ or $[c_0, b]$ (c_0 = midpoint), say $[a, c_0]$.



Then f is unbounded in $[a, c_1]$ or $[c_1, c_0]$ (c_1 = midpt.), say $[c_1, c_0]$, and so on. Call this sequence of intervals $[a_j, b_j]$, with $a_0 = a$ & $b_0 = b$; clearly $b_j - a_j = \frac{1}{2^j}(b-a)$. Write $\mathcal{A} := \{a_j \mid j \in \mathbb{Z}_{\geq 0}\} \subset [a,b]$ and $\alpha := \sup \mathcal{A} \in [a,b]$. Since $\{a_j\}$ is increasing, $\alpha = \sup\{a_j \mid j \geq n\} \in [a_n, b_n]$ for each n.

Since f is continuous at α, $\exists\, \delta > 0$ s.t.

$$x \in (\alpha - \delta, \alpha + \delta) \cap [a,b] \implies f(x) \in (f(\alpha) - 1, f(\alpha) + 1) \quad (\text{taking } \epsilon = 1)$$

$$\implies \begin{cases} f(x) < f(\alpha) + 1 \\ -f(x) < -f(\alpha) + 1 \end{cases} \xrightarrow[f \leq 0]{f \geq 0} \boxed{|f(x)| < |f(\alpha)| + 1.} \; (\dagger)$$

Taking n large enough that $\frac{1}{2^n}(b-a) < \delta$, since $\alpha \in [a_n, b_n]$ we have $[a_n, b_n] \subset (\alpha - \delta, \alpha + \delta)$ so that $(\dagger)$ bounds $f(x)$ on $[a_n, b_n]$, contradiction.

(b) <u>Claim</u>: For each $\epsilon_0 > 0$, there exists a partition P $(= \{x_0, x_1, \ldots, x_N\})$ of $[a,b]$ s.t. $\underbrace{\sup\{f(x) \mid x \in [x_{i-1}, x_i]\}}_{M_i} - \underbrace{\inf\{f(x) \mid x \in [x_{i-1}, x_i]\}}_{m_i} < \epsilon_0$ for each i.

Suppose Claim is false for $[a,b]$. Then it is false for $[a,c]$ or $[c,b]$. Arguing as above, but taking $\epsilon = \epsilon_0/4$ instead of 1, we get that $f(x) \in (f(\alpha) - \epsilon_0/4, f(\alpha) + \epsilon_0/4)$ for $x \in [a_n, b_n]$ (with n suff. large), so that $\sup - \inf \leq \epsilon_0/2 < \epsilon_0$ there, in contradiction to the failure of the Claim on $[a_n, b_n]$.

So the Claim is true and we have our partition P, which will depend on ϵ_0. Define step functions $s \le f \le t$ on $[a,b]$ with values $s_i = m_i$ & $t_i = M_i$ on $[x_{i-1}, x_i]$, so that

$$\int_a^b s(x)\,dx \le \underline{I}(f) \le \overline{I}(f) \le \int_a^b t(x)\,dx \implies$$

$$0 \le \overline{I}(f) - \underline{I}(f) \le \int_a^b t(x)\,dx - \int_a^b s(x)\,dx = \sum_{i=1}^{N} (M_i - m_i)(x_i - x_{i-1})$$

$$< \sum_{i=1}^{N} \epsilon_0 (x_i - x_{i-1}) = \epsilon_0 (b-a).$$

Since we can take ϵ_0 arbitrarily small, $\overline{I}(f) = \underline{I}(f)$. $\square$

Of course, you probably are more familiar with the Mean Value Theorem for derivatives. This will follow from the one above once we know the Fundamental Theorem of Calculus.

In contrast to the development of the integral in Apostol, the following should be familiar:

<u>Definition</u>: Assume Dom(f) contains a neighborhood of p. If $\displaystyle\lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$ exists, then we say f is <u>differentiable</u> at p and write $f'(p)$ for this limit.