Lecture 15: (Re)introduction to derivatives

In the last lecture we showed that continuous functions on a closed interval (a) are integrable and (b) actually attain their extreme and mean/average values (MVT) at some point on the interval. Today's theoretical content is mild by comparison, though we will see that (c) differentiable functions are continuous; by (a) this means they are also integrable, which sets us up nicely for the Fundamental Theorem of Calculus next week.

Definitions: (1) If Dom(f) contains a neighborhood of p, and $f'(p) := \lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$ exists, then f is said to be differentiable at p and $f'(p)$ is the derivative at p; it is said to be differentiable on (a,b) if it is differentiable at every $p \in (a,b)$, and then $f'(x)$ takes on the status of a function. (We can also talk about right & left differentiability at the endpoints.)

(2) The tangent line to the graph Γ_f at $(p, f(p))$ is defined to be the line with equation

$$y - f(p) = f'(p)\cdot(x-p).$$

(This is motivated by the picture at the end of the notes for Lecture 14.)

Theorem 1: Differentiability (at p) implies continuity (at p).

Proof: If f is differentiable at p, then $\lim_{x \to p} \frac{f(x) - f(p)}{x - p} = f'(p)$ by the limit law for compositions (i.e. substituting $h = x - p$). Choose $\delta_0 > 0$ s.t. $0 < |x - p| < \delta_0 \Rightarrow \frac{f(x) - f(p)}{x - p} \in (f'(p) - 1, f'(p) + 1)$

$\Rightarrow \left|\frac{f(x) - f(p)}{x - p}\right| < |f'(p)| + 1 =: M \Rightarrow |f(x) - f(p)| < M|x - p|.$

Now let $\epsilon > 0$ be given. Choosing $\delta := \min\{\delta_0, \frac{\epsilon}{M}\}$

$\Rightarrow |f(x) - f(p)| < M|x - p| < M\frac{\epsilon}{M} = \epsilon.$ $\square$

Notation for derivatives: take $y = f(x)$.

- Newton: $\dot{y}, \ddot{y}, \ldots$ (used in physics, esp. classical mechanics)
- Leibniz: dy/dx, d^2y/dx^2 ...
- Arbogast: Df, $D^2f, \ldots$ (used in differential equations)
- Lagrange: $f'(x)$, $f''(x), \ldots$

Differentiation rules:

(i) $(f \pm g)' = f' \pm g'$

(ii) $(fg)' = f'g + g'f$

$\rightsquigarrow (af + bg)' = af' + bg'$ if $a, b \in \mathbb{R}$

(iii) $(f/g)' = \frac{gf' - fg'}{g^2}$ (where $g \neq 0$)

Proofs: (i) is immediate from limit law for "$\pm$":

$$\lim_{h \to 0} \frac{f(x+h) \pm g(x \pm h) - (f(x) \pm g(x))}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

(iii) follows from (ii) once you know $\left(\frac{1}{g}\right)' = \frac{-g'}{g^2}$, since then

$\left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)' = \frac{f'g}{g^2} - f \cdot \frac{g'}{g^2} = \frac{f'g - fg'}{g^2}$. Now

$$\left(\frac{1}{g}\right)' = \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0} \frac{g(x) - g(x+h)}{h\, g(x+h) g(h)}$$

$$= -\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h\to 0} \frac{1}{g(x+h)g(x)} = \frac{-g'(x)}{g(x)^2}.$$

(ii): $$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h)\cdot \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x)\cdot \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h)\cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g(x) f'(x) + f(x) g'(x).$$

$\square$

Ex 1/ $f(x) = c \Rightarrow f' = 0$: $\lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$ //

Ex 2/ $f(x) = x \rightarrow f' = 1$: $\lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} 1 = 1$ //

Ex 3/ Taking Ex 2 as the base case, we show $A(n): \frac{d}{dx} x^n = n x^{n-1}$ by induction: assuming this, $\frac{d}{dx} x^{n+1} \overset{(ii)}{=} \left(\frac{d}{dx} x^n\right) x + x^n \left(\frac{d}{dx} x\right)$

$= n x^{n-1} \cdot x + x^n \cdot 1 = n x^n + x^n = (n+1) x^n \Longrightarrow A(n+1)$.

Together with (i), this gives $\frac{d}{dx} \sum_{k=0}^{n} a_k x^k = \sum_{k=1}^{n} k a_k x^{k-1}$ for polynomials, & (with (iii)) allows us to differentiate rational fcns //

Ex 4/ $f(x) = x^{1/n}$. Write $u(h) := (x+h)^{1/n}$, so that (by continuity) $\lim_{h\to 0} u(h) = u(0) = x^{1/n}$ and $u(h)^n - u(0)^n = h$. So then

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h} = \lim_{h\to 0} \frac{u(h) - u(0)}{u(h)^n - u(0)^n}$$

$(u(h) - u(0))(u(h)^{n-1} + u(h)^{n-2}u(0) + \cdots + u(0)^{n-1})$

$$= \lim_{h\to 0} \frac{1}{u(h)^{n-1} + u(h)^{n-2}u(0) + \cdots + u(0)^{n-1}} = \frac{1}{u(0)^{n-1} + u(0)^{n-2}u(0) + \cdots + u(0)^{n-1}}$$

limit laws

$$= \frac{1}{n\, u(0)^{n-1}} = \frac{1}{n x^{\frac{n-1}{n}}} = \frac{1}{n} x^{\frac{1}{n}-1}.$$

As in Ex 3, induction extends the power rule law to $x^{\frac{m}{n}}$ & hence to all rational powers. //

Ex 5/ $f(x) = \sin(x)$: recall $\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \cdot \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) = 1 \cdot \cos(x) = \cos(x).$$

$f(x) = \cos(x)$: use $\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ in the same way. Note that the way these are done in other Calculus books uses $\lim_{h\to 0} \frac{1-\cos(h)}{h} = \lim_{h\to 0} \frac{\cos(0)-\cos(h)}{h}$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{h}{2}\right)\sin\left(-\frac{h}{2}\right)}{h} = \underbrace{\lim_{h\to 0} \sin\left(\frac{h}{2}\right)}_{=\sin(0)=0} \cdot \underbrace{\lim_{h\to 0} \frac{\sin(h/2)}{h/2}}_{=1} = 0 :$$

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \sin(x) \cdot \underbrace{\lim_{h\to 0} \frac{\cos(h) - 1}{h}}_{0} + \cos(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{1} = \cos(x). \quad //$$

Ex 6/ $f(x) = \cot(x) = \frac{\cos(x)}{\sin(x)}$

By (iii), $f'(x) = \frac{\sin(x)\cos'(x) - \cos(x)\sin'(x)}{\sin^2(x)} = \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)}$

$= -\csc^2(x)$. //

Ex 7/ Derivatives are rates of change. This is reflected by the fact that the rate of change of the area of a disk w.r.t. change in radius is

$$\frac{dA}{dr} = \frac{d}{dr}\pi r^2 = 2\pi r = \text{circumference},$$

and the rate of change of the volume of a ball is

$$\frac{dV}{dr} = \frac{d}{dr}\frac{4}{3}\pi r^3 = 4\pi r^2 = \text{surface area}. \quad //$$

Ex 8/ Claim: $y = -x$ is tangent to $y = x^3 - 6x^2 + 8x$.

To find intersection points, write $-x = x^3 - 6x^2 + 8x$

$\rightsquigarrow 0 = x^3 - 6x^2 + 9x = x(x-3)^2 \rightsquigarrow x = 0, 3$

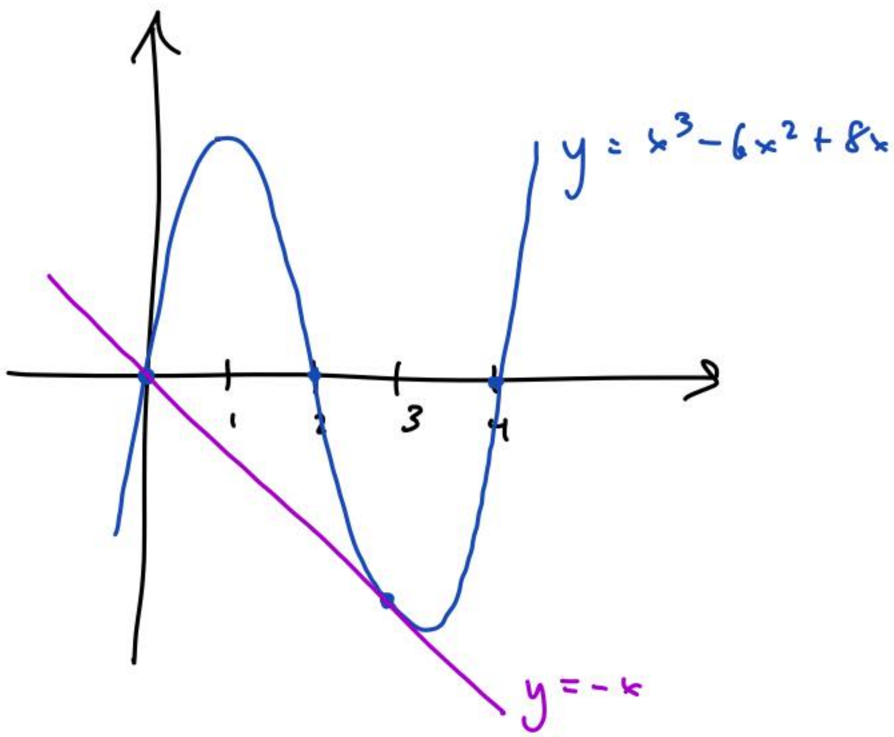
$\rightsquigarrow (x,y) = (0,0)$ & $(3,-3)$.

The line has slope -1. The curve has slope

$f'(x) = 3x^2 - 12x + 8$ which is 8 at $(0,0)$

and ... -1 at $(3,-3)$. So indeed, the line

is tangent to the curve at $(3,-3)$. In a picture:



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