Lecture 17: The Fundamental Theorems

Main Point of today's lecture:

- taking the "slope" function ($f(x) \rightsquigarrow f'(x)$)

and

- taking the "area-up-to-x" function ($f(x) \rightsquigarrow \int_c^x f(t)\,dt$)

are inverse operations.

Let I be an open interval.

Definition: $P : I \to \mathbb{R}$ is a primitive (or antiderivative) of $f : I \to \mathbb{R}$ if $P' = f$ on I.

FTC I ("Indefinite integrals yield primitives")

Assume: f integrable on $[a,x]$ $\forall x \in [a,b]$, and $c \in [a,b]$.

(e.g. f is bounded + piecewise monotonic on $[a,b]$, or f is piecewise continuous on $[a,b]$)

Set $A(x) := \int_c^x f(t)\,dt$, $\mathcal{S}_f := \{x \in (a,b) \mid f \text{ is continuous at } x\}$.

Then: $A'(x)$ exists and equals $f(x)$ at every $x \in \mathcal{S}_f$.

FTC II. Assume: f continuous on I, and $c \in I$, $[a,b] \subset I$. Then:

$\boxed{\text{v. 1}}$ ("Every primitive is an indefinite integral")

$$P(x) = P(c) + \int_c^x f(t)\,dt$$

$\longrightarrow$ (just substitute $x = b$, $c = a$)

$\boxed{\text{v. 2}}$ ("Definite integral = change in primitive")

$$\int_a^b f(x)\,dx = P(b) - P(a) = P(x)\Big|_a^b.$$

You've probably all had enough of using FTC II to compute integrals, but here is one we weren't able to do from the definition of the integral: fractional powers.

Ex 1/ We showed $\frac{d}{dx} x^{q+1} = (q+1)x^q$ for any $q \in \mathbb{Q}$. So $\frac{x^{q+1}}{q+1}$ is a primitive for x^q so long as $q \neq -1$, and (for $0 \leq a \leq b$)

$$\int_a^b x^q \, dx = \frac{b^{q+1} - a^{q+1}}{q+1} \quad \text{by FTC II.} \quad //$$

Ex 2/ If $q = -1$, define (for $x > 0$) $\log(x) := \int_1^x \frac{1}{t} \, dt$.

By FTC I, $\frac{d}{dx} \log(x) = \frac{1}{x}$. //

Ex 3/ The FTCs elucidate the relationship between the MVTs. Suppose we only knew the one for derivatives, and let f be continuous on $[a,b]$. Then f is integrable there and

$$\int_c^x f(t)\,dt =: F(x) \text{ is } \begin{cases} \text{continuous on } [a,b] & \text{(by Lecture 12)} \\ \text{differentiable on } (a,b), \text{ with } F' = f & \text{(by FTC I)}. \end{cases}$$

So F satisfies the hypotheses of the MVT for derivatives, and $\exists\, c \in (a,b)$ s.t. $F'(c) = \frac{1}{b-a}(F(b) - F(a))$. Since $F' = f$ & $F(b) - F(a) = \int_a^b f(x)\,dx$ (FTC II), this reads $f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt$.

If g is also continuous on $[a,b]$, with primitive G, and fg has primitive H, then applying the Cauchy MVT to H & G gives $\frac{H'(c)}{G'(c)} = \frac{H(b) - H(a)}{G(b) - G(a)}$ hence $f(c) = \frac{\int_a^b f(x) g(x)\,dx}{\int_a^b g(x)\,dx}$, which is the weighted MVT for integrals!

Ex 4/ $p(t)$ = (vertical) position of ball

$v(t) = p'(t)$ = velocity

$a(t) = v'(t) = p''(t)$ = acceleration.

If $a = a_0$ is constant, then $v(t) = a_0 t + v_0$ $(v_0 = v(0))$, and the change in position is $p(T) - p(0) = \int_0^T (a_0 t + v_0)\,dt = a_0 \frac{T^2}{2} + v_0 T$.

For instance, if the ball starts from rest $(v_0 = 0)$, and $a_0 = -g$, (accel due to gravity)

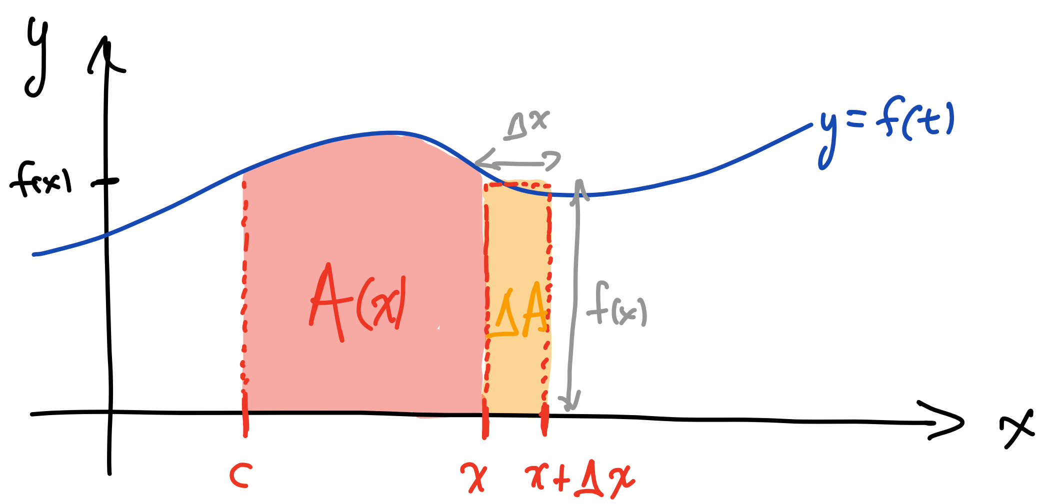
then $p(T) - p(0) = -\frac{g}{2} T^2$.

But if we factor in air-resistance, instead of $p'' = -g$ we have to "integrate" $p'' = -g - \kappa p'$, which is a full-fledged differential equation. (Can't solve this yet!) //

Proofs. The key point is that the rate of accumulation of area under the graph of a function is proportional to its height.

Set

$$A(x) := \int_c^x f(t)\,dt.$$



Heuristically

$$\frac{\Delta A}{\Delta x} \approx \frac{f(x)\cdot \Delta x}{\Delta x} = f(x).$$

All we need to do is to make this a bit more precise.

Proof of FTC I: We must show that $A'(x)$ exists & $= f(x)$ provided f is continuous at x.

Compute
$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = \lim_{h\to 0} \frac{\int_c^{x+h} f(t)\,dt - \int_c^{x} f(t)\,dt}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

To show this limit exists, let $\epsilon > 0$: by continuity of f at x, $\exists\, \delta > 0$ s.t.
$$t \in (x-\delta, x+\delta) \implies \boxed{f(x) - \frac{\epsilon}{2} < f(t) < f(x) + \frac{\epsilon}{2}}. \quad (*)$$

Taking $0 < h < \delta$, $[x, x+h] \subset (x-\delta, x+\delta)$ and so $(*)$ holds on the interval of integration. By the comparison property,
$$\frac{1}{h}\int_x^{x+h} \underbrace{\left(f(x) - \frac{\epsilon}{2}\right)}_{\text{constant in } t} dt \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le \frac{1}{h}\int_x^{x+h} \underbrace{\left(f(x) + \frac{\epsilon}{2}\right)}_{\text{constant in } t} dt$$
$$\frac{1}{h}\int_x^{x+h} \left(f(x) - \frac{\epsilon}{2}\right) dt = f(x) - \frac{\epsilon}{2}, \qquad \frac{1}{h}\int_x^{x+h} \left(f(x) + \frac{\epsilon}{2}\right) dt = f(x) + \frac{\epsilon}{2}$$

hence
$$\left| \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x) \right| \le \frac{\epsilon}{2} < \epsilon.$$
This gives
$$\lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x),$$
and a similar analysis (reversing inequalities) gives the $\lim_{h\to 0^-}$. $\square$

Proof of FTC II: Define $A(x) := \int_c^x f(t)\,dt$. By FTC I & continuity of f, $A'(x) = f(x)$. So P, A are both primitives
$$\Rightarrow (P-A)' = P' - A' = f - f = 0 \Rightarrow P - A = k \text{ (constant)}$$
$$\xrightarrow[\text{at } x = c]{\text{evaluate}} k = P(c) - \cancelto{0}{A(c)} = P(c) \Rightarrow P(x) = P(c) + A(x). \quad \square$$