Lecture 18 : Chain rule & substitution

Suppose that $y = F(u)$ & $u = g(x)$ determine a composite function $(F \circ g)(x) := F(g(x))$ $(= y)$ on an open interval I.

i.e. for those $x \in I$ at which ...

<u>Theorem 1 (Chain rule)</u> : Wherever $F'(g(x))$ and $g'(x)$ both exist,

$(F \circ g)'(x)$ exists and equals $F'(g(x)) \cdot g'(x)$.

Ways to think about this:

- as functions : $(F \circ g)' = (F' \circ g) \cdot g'$
- in Leibniz notation : $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- in words : when you compose functions, their rates of change multiply.

$$\begin{array}{l} y \\ \big| \ {}^{dy/du} \\ u \\ \big| \ {}^{du/dx} \\ x \end{array}$$

<u>Proof</u> : Fix an $x \in I$ s.t. the hypothesis of the Thm. holds.

Define (for $t \in \mathcal{N}(0)$)

$$q(t) := \begin{cases} \dfrac{F(g(x)+t) - F(g(x))}{t}, & t \neq 0 \\ F'(g(x)), & t = 0 \end{cases}$$

which is continuous at 0 since F is diff'able at $g(x)$. Clearly

$$(*) \qquad t\, q(t) = F(g(x)+t) - F(g(x))$$

holds in $\mathcal{N}(0)$.

Note that $k(h) := g(x+h) - g(x)$ is continuous at

$h = 0$ (with value 0) since g is diff. $\Rightarrow$ cts. at x. So for h in a smaller nbhd. $\mathcal{N}_1(0)$ we have $k(h) \in \mathcal{N}(0)$, and substituting $t = k(h)$ in $(*)$ & dividing by h gives

$$\frac{k(h)}{h} q(k(h)) = \frac{F(g(x) + k(h)) - F(g(x))}{h}$$

for $h \in \mathcal{N}_1^*(0)$ (punctured nbhd.), i.e.

$$\frac{g(x+h) - g(x)}{h} \cdot q(k(h)) = \frac{F(g(x+h)) - F(g(x))}{h}.$$

Taking $h \to 0$ limits on both sides gives the result, since
$\lim_{h\to 0} q(k(h)) = q\left(\lim_{h\to 0} k(h)\right) = q(0) = F'(g(x))$. $\square$

Ex 1/ $\underbrace{f(x) = \sqrt{1+x^2}}_{F(g(x))} \Rightarrow f'(x) = \underbrace{\frac{1}{2\sqrt{1+x^2}}}_{F'(g(x))} \cdot \underbrace{2x}_{g'(x)}$ //

Ex 2/ $\frac{d}{dx} \underbrace{\int_0^{\sin(x)} f(t)\, dt}_{F(g(x))} = \underbrace{f(\sin(x))}_{F'(g(x))} \cdot \underbrace{\cos(x)}_{g'(x)}$. //

Ex 3/ $f(x) = \underbrace{\sin(\overbrace{\sin(\underbrace{\sin(x)}_{v})}^{u})}_{y} \Rightarrow f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$

$= \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x)$. //

Ex 4/ $x^2 + y^2 = r^2$ gives y <u>implicitly</u> rather than <u>explicitly</u> $(= f(x))$ in terms of x. Since it is secretly a function of x, you must use the chain rule and write $\frac{dy^2}{dx} = \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 2y \cdot y'$, hence $2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y} \Rightarrow$ tangent to a circle is perpendicular to the radius (why?). //

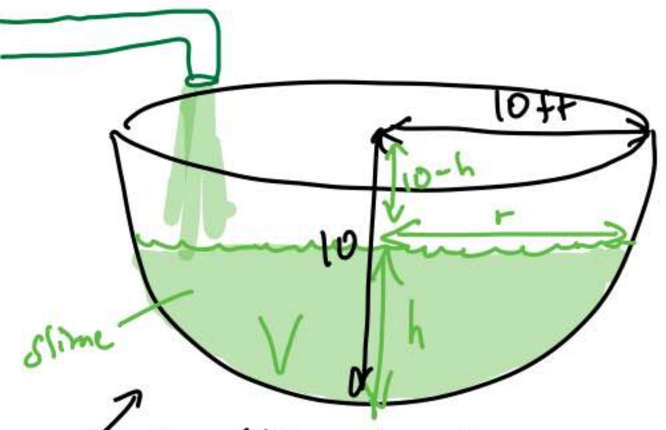
Ex 5/ (More implicit diff.)

$$x^3+y^3=1 \xrightarrow[d/dx]{} \cancel{3}x^2+\cancel{3}y^2y'=0 \xrightarrow[d/dx]{} 2x+2y(y')^2+y^2y''=0$$

$$\xrightarrow[\cdot y^3]{} 2xy^3+2\underbrace{(y^2y')^2}_{-x^2}+y^5y''=0 \to 2x\underbrace{\cancel{(y^3+x^3)}}_{1}+y^5y''=0$$

$$\to y''=-2xy^{-5}. \qquad //$$

Ex 6/ (A related rates problem)



backyard pool = hemisphere

(V = volume)

$$r(h)=\sqrt{10^2-(10-h)^2}$$
$$=\sqrt{20h-h^2}$$

- Calculate $\frac{dV}{dh}$ when $h=5$:

$$V=\int_0^h A(h)\,dh \quad \leftarrow \text{cross-sectional area function}$$

$$\frac{dV}{dh}=A(h)=\pi(r(h))^2=\pi(20h-h^2)$$

$$\left.\frac{dV}{dh}\right|_{h=5}=\pi(100-25)=75\pi \ \text{ft}^3/\text{ft}$$

- If $\frac{dV}{dt}=5\sqrt{3}$ (constant), find $\frac{dr}{dt}$ when $h=5$.

$$\frac{dr}{dt}=\frac{dr}{dh}\cdot\frac{dh}{dV}\cdot\frac{dV}{dt}=\frac{r'(h)\cdot 5\sqrt{3}}{dV/dh}, \quad \text{where } r'(h)=\frac{10-h}{\sqrt{20h-h^2}}. \quad \text{So}$$

$$r'(5)=\frac{5}{\sqrt{75}}=\frac{1}{\sqrt{3}} \quad \text{and} \quad \left.\frac{dr}{dt}\right|_{h=5}=\frac{\frac{1}{\sqrt{3}}\cdot 5\sqrt{3}}{75\pi}=\frac{1}{15\pi}\ \text{ft}/\text{s}. \qquad //$$

Now, the entire reason I waited until now to do the chain rule was so that we could also do the "inverse of the chain rule" for integrals, otherwise known as substitution.

<u>Theorem 2</u>: If g' is continuous on I, f is continuous on $g(I)$, and $x, c \in I$, then

$$\mathcal{J}(x) := \int_c^x f(g(t))\, g'(t)\, dt = \int_{g(c)}^{g(x)} f(u)\, du .$$

<u>Proof</u>: Set $F(w) := \int_{g(c)}^{w} f(u)\, du$, so then $F'(w) = f(w)$.

By the Chain rule, $\mathcal{J}'(x) = f(g(x))\, g'(x) = F'(g(x))\, g'(x) = (F \circ g)'(x)$.

Hence $\mathcal{J} - F \circ g = K$ (constant), and evaluating at c gives

$K = \mathcal{J}(c) - F(g(c)) = 0 - 0 = 0$. So $\mathcal{J}(x) = F(g(x))$. $\square$

Ex/ $$\int_0^{\pi/3} \frac{\sin(x)}{\sqrt{\cos^3(x)}}\, dx = \int_{\cos(0)}^{\cos(\pi/3)} \frac{-du}{\sqrt{u^3}} = -\int_1^{1/2} u^{-3/2}\, du$$

$u = \cos(x)$

$du = -\sin(x)\, dx$

$$= -\frac{u^{-3/2+1}}{-3/2+1}\Bigg|_1^{1/2} = 2u^{-1/2}\Big|_1^{1/2} = 2\sqrt{2} - 2 .$$ //