Lecture 19: Integration by Parts

We conclude our discussion of the relationship between $\int dx$ and $\frac{d}{dx}$ with one more idea. Let $u(x), v(x)$ have continuous derivatives.

Product Rule: gives derivative of their product

$$u'(x)\,v(x) + u(x)\,v'(x) = (u(x)v(x))'.$$

We successfully "reversed" the Chain Rule to get substitution for integrals. When we try to "reverse" the Product Rule, we get the so-called integration by parts:

$$\int \left(u'(x)v(x) + u(x)v'(x)\right) dx = u(x)\cdot v(x) + C$$

$$\Rightarrow \int u'(x)\,v(x)\,dx + \int u(x)\,v'(x)\,dx = u(x)\,v(x) + C$$

$$\Rightarrow (*) \quad \boxed{\int u(x)\,v'(x)\,dx = u(x)\,v(x) - \int u'(x)\,v(x)\,dx,}$$

absorbed into here

or more compactly

$$\int u v'\,dx = uv - \int v u'\,dx.$$

By using the shorthand $\begin{cases} du = \frac{du}{dx}dx = u'\,dx \\ dv = \frac{dv}{dx}dx = v'\,dx \end{cases}$

we obtain

$$\boxed{\int u\,dv = uv - \int v\,du} \quad \left(\int \text{ by parts (I)}\right)$$

(up to a constant)

The trick in applying this method is in deciding which part of the original integral should be u and which dv.

Ex 1/ $$\int x \log x \, dx = \frac{x^2}{2}\log(x) - \int \frac{x^2}{2}\frac{dx}{x}$$

$$\left[\begin{matrix} u = \log(x), & dv = x\,dx \\ du = \frac{dx}{x}, & v = x^2/2 \end{matrix} \right]$$

we know nothing about this function, except that it is def'd. on $\mathbb{R}^+$ and $\log'(x) = \frac{1}{x}$

$$= \frac{x^2}{2}\log(x) - \int \frac{x}{2}\,dx = \frac{x^2}{2}\log(x) - \frac{x^2}{4} + C.$$

$$\left[\text{Check: } \frac{d}{dx}\left[\frac{x^2}{2}\log(x) - \frac{x^2}{4}\right] = x\log(x) + \cancel{\frac{x^2}{2}\frac{1}{x} - \frac{x}{2}} = x\log(x).\right]$$

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Ex 2/ $$\int \log(x)\,dx = x\log(x) - \int x\,\frac{dx}{x}$$

$$\left[\begin{matrix} u = \log(x), & dv = dx \\ du = dx/x, & v = x \end{matrix} \right] \qquad = x\log(x) - \int dx$$

$$= x\log(x) - x + C.$$

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MAIN POINTS: ① The dv needs to be something that you already know how to integrate.

② $\int v\,du$ had better be easier than $\int u\,dv$!

Otherwise - choose different u, dv!

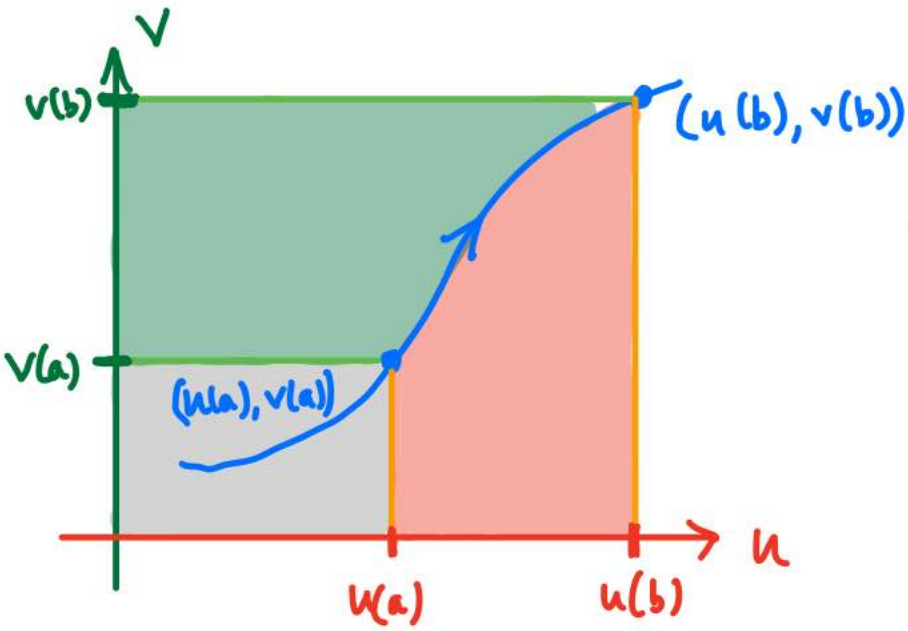
Now indefinite integrals are just primitives, so we can evaluate (*) at b and at a, then subtract, to get

$$\int u(x)\,v'(x)\,dx \Big|_a^b = u(x)\,v(x)\Big|_a^b - \int v(x)\,u'(x)\,dx\Big|_a^b$$

$$\Longrightarrow \int_a^b u(x)\,v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)\,u'(x)\,dx.$$

But there is also a more geometric way to derive this in the case where $u(x)$ and $v(x)$ are strictly increasing.

Let $x \longmapsto (u(x), v(x))$ parametrize a curve



Think of x as time, so that u & v are the coordinates of an airplane, which starts at $(u(a), v(a))$ & ends at $(u(b), v(b))$. Having drawn the curve, we can now also think of it as expressing $\underbrace{u \text{ as a function of } v}_{u(v)}$, $\underbrace{\text{or vice versa}}_{v(u)}$.

Now consider the area of the big rectangle, which is the sum of areas of the three shaded regions:

$$\int_{u(a)}^{u(b)} v(u)\, du + \int_{v(a)}^{v(b)} u(v)\, dv + u(a)v(a) = u(b)v(b)$$

$$\Rightarrow \int_{v(a)}^{v(b)} u\, dv = uv\Big|_a^b - \int_{u(a)}^{u(b)} v\, du.$$

Now if you interpret $\begin{cases} dv \text{ as } v'\,dx \\ du \text{ as } u'\,dx \end{cases}$, so that the integral is actually over x (as is the case in practice), then x goes from a to b:

$$\boxed{\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du.} \quad \left(\int \text{ by parts (II)}\right)$$

(with the caveat that the integrals being performed are $\int_a^b uv'dx$, $\int_a^b vu'dx$).

Ex 3/ $\displaystyle\int_0^{\pi/6} x\cos(x)\,dx = x\sin(x)\Big|_0^{\pi/6} - \int_0^{\pi/6}\sin(x)\,dx$

$$\begin{bmatrix} u = x, & dv = \cos(x)\,dx \\ du = dx, & v = \sin(x) \end{bmatrix}$$

Why not $u = \cos(x)$ and $dv = x\,dx$? b/c $v = \frac{x^2}{2}$ then makes it worse!

$$= \frac{\pi}{6}\cdot\frac{1}{2} - 0\cdot 0 + \underbrace{\cos(x)\Big|_0^{\pi/6}}_{\sqrt{3}/2\, -\, 1} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1. \quad //$$

Ex 4/ $\displaystyle\int \frac{1}{x}\,dx = \frac{1}{x}\,x - \int x\,\frac{-dx}{x^2} = 1 + \int\frac{1}{x}\,dx$

$$\begin{bmatrix} u = \frac{1}{x} & dv = dx \\ du = \frac{-dx}{x^2} & v = x \end{bmatrix} \qquad \Rightarrow \quad 1 = 0. \text{ (Right?)} \quad //$$

[Of course, point is that primitives written as $\int \cdots dx$ include an often unwritten arbitrary constant.]

Some problems also require repeated integration by parts; the book contains an example.

We are skipping §4.22 on partial derivatives for now, though you should look it over (and maybe I'll discuss it briefly).

Finally, though Apostol happily ignores this, it is true that if $f \le g$ on an interval $[a,b]$ (where f & g are both integrable), and $f(c) < g(c)$ for some point $c \in [a,b]$ at which $g-f$ is continuous, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$. Just apply p. 155 #7 to $g-f$!

So in particular, if f & g are piecewise continuous, and $f < g$, then $\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$.