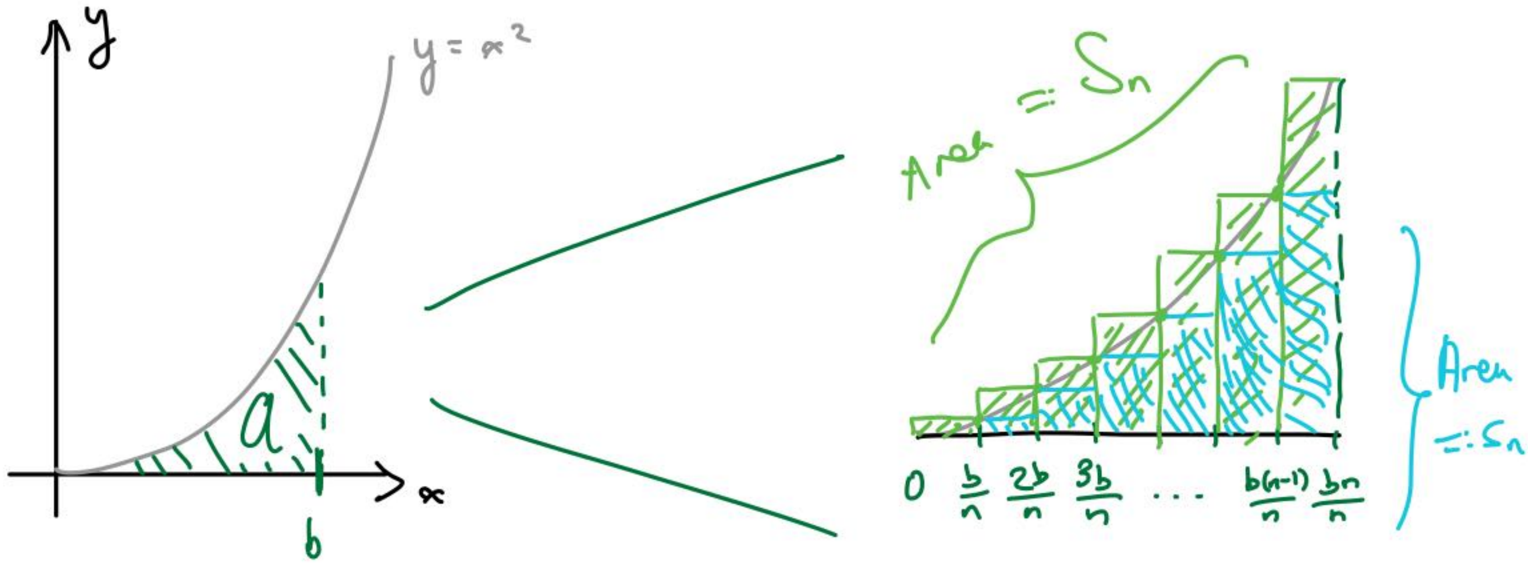
Lecture 2: Area of a parabolic segment

Around a quarter of a millenium B.C., Archimedes managed to compute what we would now recognize as a definite integral. We begin with Apostol's "modern" version of his calculation, which is really only a special case.



For any positive integer n, we may consider the areas of the "unions of boxes" above and below the curve $y = x^2$:

$$S_n = \sum_{k=1}^{n} \frac{b}{n}\left(\frac{kb}{n}\right)^2 = \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 = \frac{b^3}{n^3} P_2(n)$$

and

$$s_n = \sum_{k=0}^{n-1} \frac{b}{n}\left(\frac{kb}{n}\right)^2 = \frac{b^3}{n^3}\sum_{k=1}^{n-1} k^2 = \frac{b^3}{n^3} P_2(n-1)$$

why?

Our intuition regarding areas (when one region contains another) dictates that

$$s_n \leq a \leq S_n \qquad \text{for all } n \geq 1.$$

Now (**) from Lecture 1 says that

$$P_2(n-1) < \frac{n^3}{3} < P_2(n) \quad (\text{for all } n \geq 1),$$

or (multiplying by $\frac{b^3}{n^3}$)

$$\left(s_n =\right) \frac{b^3}{n^3} P_2(n-1) < \frac{b^3}{3} < \frac{b^3}{n^3} P_2(n) \left(= S_n\right).$$

Suppose $A \neq \frac{b^3}{3}$, so that $\varepsilon := \left|A - \frac{b^3}{3}\right| > \frac{1}{N}$ for some positive integer N. Since both A & $b^3/3$ are sandwiched between s_n & S_n, we get

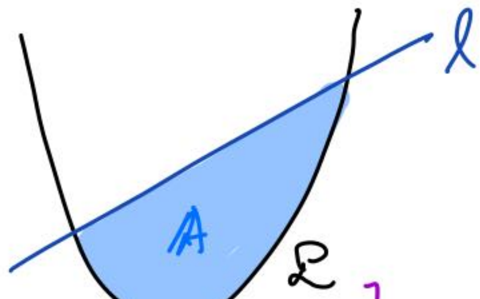
$$\frac{1}{N} < \varepsilon < S_n - s_n = \frac{b^3}{n^3}\underbrace{\Big(P_2(n) - P_2(n-1)\Big)}_{n^2} = \frac{b^3}{n}$$

for all n. But this is false as soon as $n \geq b^3 N$. So $\boxed{A = \frac{b^3}{3}}$.

[Remark: If we had defined limits, we could just plug in our formulas for $P_2(n-1)$ & $P_2(n)$ & use the squeeze theorem. But we don't yet have the concept of "limit". (Sorry.)]

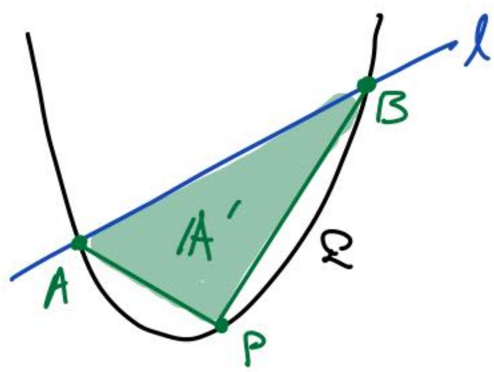
But this isn't remotely what Archimedes did.

Let A be the area of the parabolic segment bounded by a line ℓ and a parabola $\mathcal{L}$:



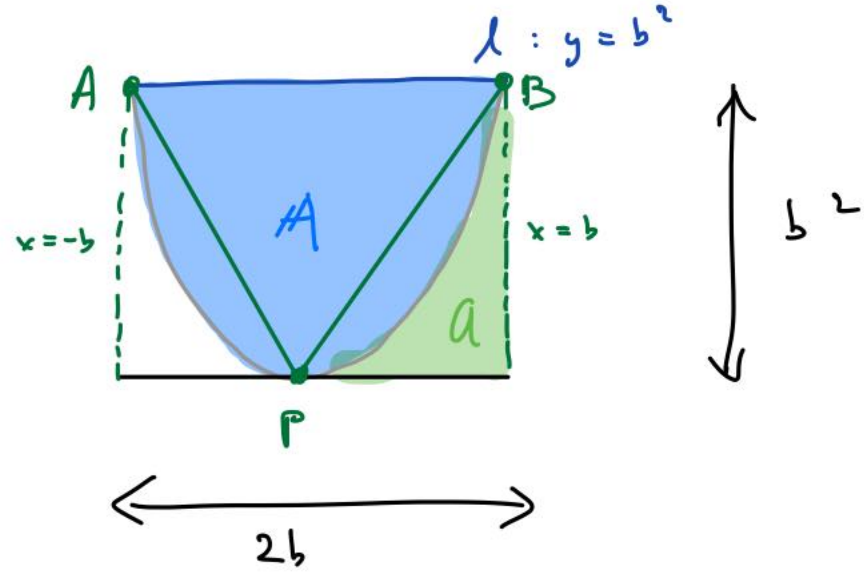
[The original source:
http://www.math.ubc.ca/~cass/courses/m309-8a/java/images/archimedes/parabola.html]

and $\mathbb{A}'$ the area of the triangle APB (where P is the point on $\mathcal{P}$ farthest from ℓ):



<u>Archimedes's Theorem</u>: $\mathbb{A} = \frac{4}{3}\mathbb{A}'$.

This is much more general than, but immediately implies, the result in Apostol: for consider the picture



We have

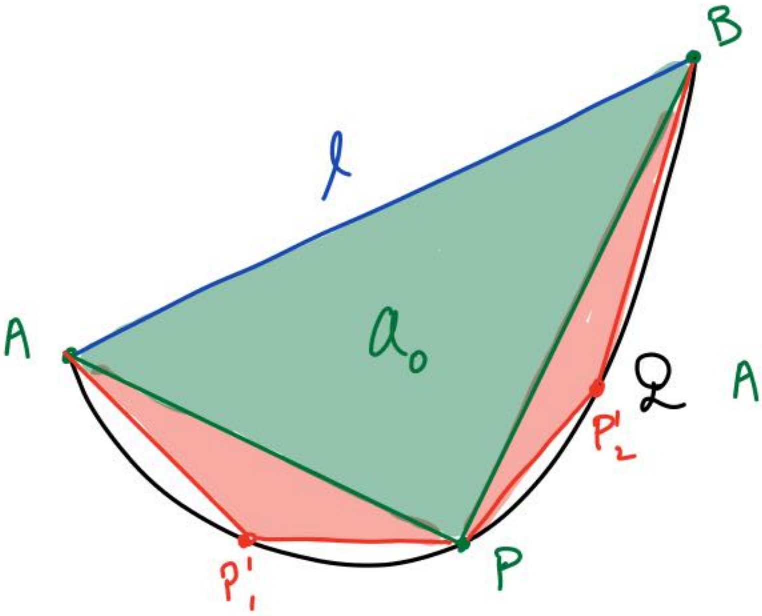
$$\text{area (box)} = \mathbb{A} + 2a$$

$$2b \cdot b^2 = \frac{4}{3} \cdot \underbrace{\text{area}(APB)}_{\frac{1}{2}\cdot 2b \cdot b^2} + 2a$$

$$2b^3 = \frac{4}{3} b^3 + 2a$$

$$\Rightarrow \quad a = \frac{1}{3} b^3, \quad \text{as desired.}$$

So how did Archimedes prove this more general theorem?



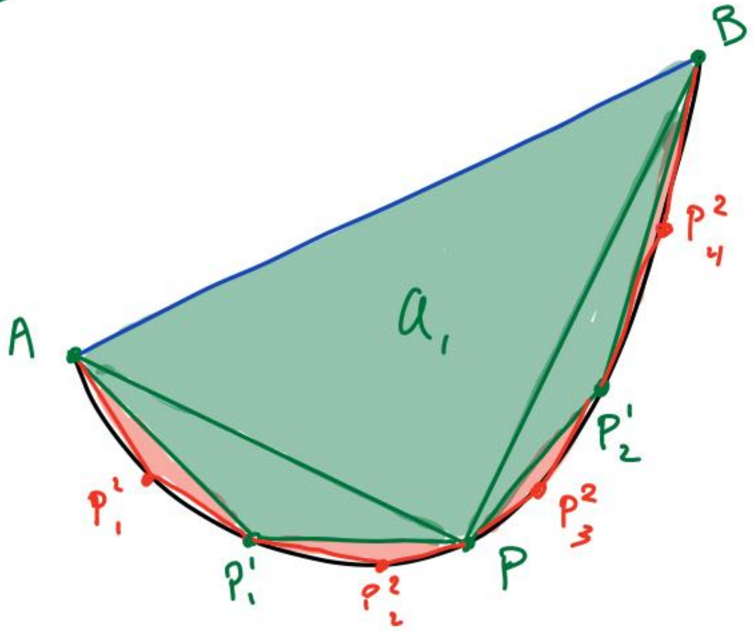
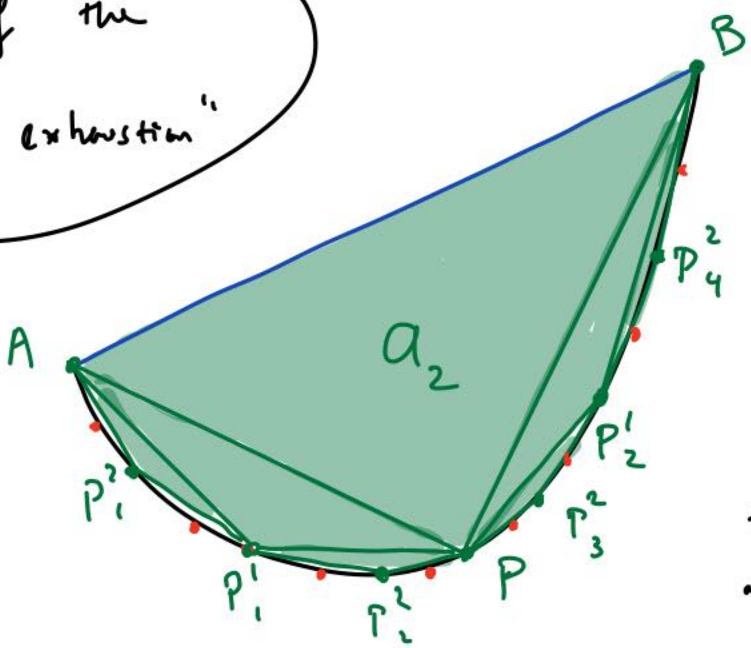


Illustration of the "method of exhaustion"

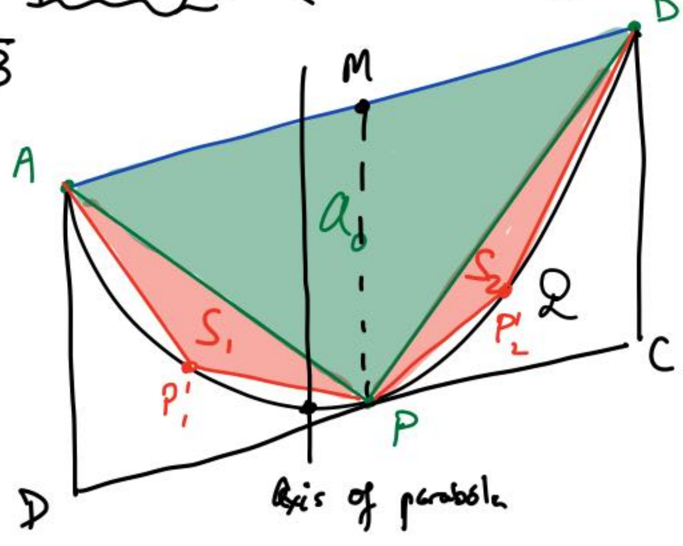


Write

$$\varepsilon_j := A - a_j$$

for the "leftover" area at each step.

We shall use several results known in Archimedes's time without proof: First, we have a parallelogram (with area $2a_0$) in which M is the midpoint of $\overline{AB}$ and $\overline{DC}$ is tangent to $\mathcal{Q}$ at P:



$$A < 2a_0 \Rightarrow \frac{A}{2} < a_0$$

$$\Longrightarrow \varepsilon_0 = A - a_0 < \frac{1}{2} A.$$

By the same token, if

$\mathcal{S}_i$ is the parabolic segment approximated by S_i, then we deduce $\mathcal{S}_i - S_i < \frac{1}{2}\mathcal{S}_i$, and then

$$\begin{aligned}\varepsilon_1 = \mathbb{A} - a_1 &= (a_0 + \mathcal{S}_1 + \mathcal{S}_2) - (a_0 + S_1 + S_2) \\ &= (\mathcal{S}_1 - S_1) + (\mathcal{S}_2 - S_2) < \tfrac{1}{2}(\mathcal{S}_1 + \mathcal{S}_2) = \tfrac{1}{2}\varepsilon_0 .\end{aligned}$$

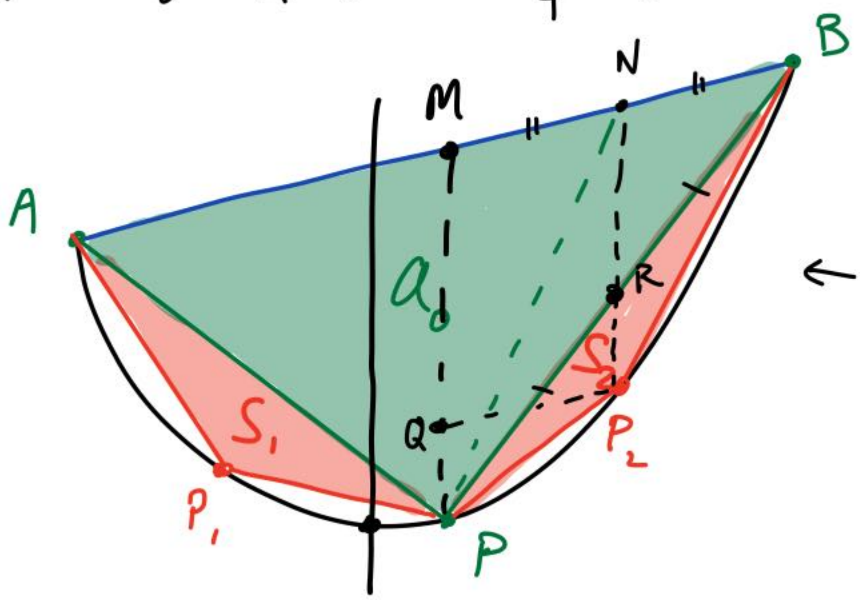
Continuing in this fashion gives $\varepsilon_j < \frac{1}{2}\varepsilon_{j-1}$ (for each j) hence $\varepsilon_j < \frac{1}{2^{j+1}}\mathbb{A}$. (So these "errors" go to zero.)

Next, we claim that

$$(\dagger) \qquad \boxed{S_1 + S_2 = \tfrac{1}{4} a_0},$$

so that $a_1 = a_0 + \frac{1}{4}a_0$, and (continuing the process)

$$\left.\begin{aligned} a_2 &= a_0 + \tfrac{1}{4}a_0 + \tfrac{1}{4^2}a_0 \\ &\vdots \\ a_n &= a_0 + \tfrac{1}{4}a_0 + \cdots + \tfrac{1}{4^n}a_0 \end{aligned}\right\} \text{(why?)}$$



← Notice that by similar triangles $PM = 2RN$

Another key property known by the ancient Greeks is that

$$\frac{PQ}{PM} = \frac{(QP_2)^2}{MB^2} = \frac{MN^2}{(2\,MN)^2} = \frac{1}{4}$$

(basically says that the parabola still looks like $y = x^2$ in the oblique coordinate system with axes given by the parallelogram).

So we have $PM = 4PQ$, which implies $P_2N = 3PQ$,

hence $\begin{cases} RN = \frac{1}{2}PM = 2PQ \\ P_2R = P_2N - RN = 3PQ - 2PQ = PQ \end{cases} \Longrightarrow P_2R = \frac{1}{2}RN$

$$\Longrightarrow S_2 = \text{area}(PP_2B) = \frac{1}{2}\text{area}(PNB) = \frac{1}{4}\text{area}(PMB).$$

Similarly, $S_1 = \frac{1}{4}\text{area}(PMA)$, and so

$$S_1 + S_2 = \frac{1}{4}\text{area}(PAB) = \frac{1}{4}a_0, \quad \text{as desired.}$$

Finally, we want to show $\mathcal{A} = \frac{4}{3}a_0$. (formerly $\mathcal{A}'$)

We know that $0 \le \underbrace{\mathcal{A} - a_n}_{\varepsilon_n} < \frac{1}{2^{n+1}}\mathcal{A}$ and that

(*) from Lecture 1

$$a_n = \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^n}\right)a_0 = S_{\frac{1}{4}}(n)\,a_0 = \frac{4}{3}a_0 - \frac{1}{3}\frac{1}{4^n}a_0.$$

That is, $\quad 0 \le \mathcal{A} - \frac{4}{3}a_0 + \frac{1}{3}\cdot\frac{1}{4^n}a_0 < \frac{1}{2^{n+1}}\mathcal{A}$

$$\Rightarrow -\frac{1}{3}\cdot\frac{1}{4^n}a_0 \le \mathcal{A} - \frac{4}{3}a_0 < \underbrace{\frac{1}{2^{n+1}}\mathcal{A} - \frac{1}{3\cdot 4^n}a_0}_{\text{positive}}.$$

If the middle quantity were positive, you'd get a contradiction to the "<" by taking n large enough. Were it negative, you'd get a contradiction to the "≤".

So $\mathcal{A} - \frac{4}{3}a_0 = 0$, done. (Again, it's all much easier once we have limits, which Archimedes didn't:

$$\mathcal{A} = \lim_{n\to\infty} a_n = \lim_{n\to\infty} P_{\frac{1}{4}}(n)\,a_0 = \lim_{n\to\infty}\frac{1-\frac{1}{4^{n+1}}}{1-\frac{1}{4}}\,a_0 = \frac{1}{3/4}a_0 = \frac{4}{3}a_0.)$$

Well! That was hard. The rest of the week will be back to foundations — sets, real numbers, and induction.