Lecture 20: The logarithm

Definition: $\log(x) := \int_1^x \frac{dt}{t}$ for $x \in \mathbb{R}^+$.

Let's deduce some properties from this:

(1) $\log(1) = 0$

(2) $\log$ is differentiable & (thus) continuous on $\mathbb{R}^+$, by FTC

(3) $\log'(x) = \frac{1}{x}$

(4) $\log(wx) = \log(w) + \log(x)$ $\forall x, w \in \mathbb{R}^+$

Proof: $\int_1^{wx} \frac{dt}{t} = \int_1^{w} \frac{dt}{t} + \int_w^{wx} \frac{dt}{t} \overset{t = wu}{=} \log(w) + \int_1^x \frac{du}{u} = \log(w) + \log(x)$. $\square$

(5) $\log\left(\frac{1}{w}\right) = -\log(w)$ (take $x = \frac{1}{w}$ in (4))

(6) $\log(w^r) = r\log(w)$ for any $r \in \mathbb{Q}$

Proof: Inductively one has for $n \in \mathbb{P}$ $\log(z^n) = n\log(z)$. Taking $z = w^{1/n}$ $\Rightarrow \frac{1}{n}\log(w) = \log(w^{1/n}) \Rightarrow \log(w^{m/n}) = \frac{m}{n}\log(w)$. Finally combine with (5). $\square$

(7) $\log$ is strictly increasing (since $\log' > 0$)

(8) $\log$ is strictly concave (since $\log'' < 0$)

(9) $\log$ is neither bounded above nor bounded below

Proof: Given $M \in \mathbb{R}^+$, Archimedean property $\Rightarrow \exists\, n \in \mathbb{P}$ s.t. $n\log 2 > M$ $\Rightarrow \log 2^n > M$ & $\log 2^{-n} < -M$. $\square$

(10) log has a strictly increasing, continuous inverse defined on all of $\mathbb{R}$; call this <u>exp</u>. Define $e := \exp(1) \in \mathbb{R}^+$.

<u>Proof</u>: Given $y_0 \in \mathbb{R}$, choose (by (9)) $n \in \mathbb{P}$ s.t. $y_0 \in [\log(2^{-n}), \log 2^n]$.
Intermediate Value Thm. $\Rightarrow \exists\, x_0 \in [2^{-n}, 2^n]$ s.t. $\log(x_0) = y_0$.
If $\log(x_1) = y_0$ also, then $x_1 > x_0$ & $x_1 < x_0$ both give contradictions (use (7)); so x_0 is unique, and we can set $\exp(y_0) = x_0$.
For continuity, see the end of lecture 13. $\square$

(11) $\log(e) = 1$.

<u>Definition</u>: For any $b \in \mathbb{R}^+$, $\log_b(x) := \dfrac{\log(x)}{\log(b)}$.

This has $\log_b(b) = 1$ and satisfies properties (1) – (10), except replacing (3) by $\log_b'(x) = \dfrac{1}{x \log(b)}$, and $\log_b'(1) = \dfrac{1}{\log(b)}$ in particular.

It is interesting to ask whether the family of functions $\{\log_b \mid b \in \mathbb{R}^+\}$ (and the zero-function) are the <u>only</u> functions defined on all of $\mathbb{R}^+$ which satisfy the <u>functional equation</u>

$$(*) \qquad \log(wx) = \log(w) + \log(x).$$

Actually, the answer is "no": there are uncountably many pathological, everywhere-discontinuous functions satisfying (*).
But if we limit ourselves to continuous functions, or monotonic functions, the answer is "yes":

Theorem: Suppose $f: \mathbb{R}^+ \to \mathbb{R}$ satisfies $(*)$, & is either strictly monotonic or continuous. Then f is either 0 or $\log_b$ for some $b \in \mathbb{R}^+$.

Proof: To begin, notice that $(*) \Rightarrow f(1) = f(1) + f(1) \Rightarrow f(1) = 0$, as well as properties (5) - (6).

First, suppose f is strictly monotonic. We claim that f is continuous. We may replace f by $-f$, if necessary, to make f strictly increasing. It suffices to show f continuous at 1, since then $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} f(x/x_0) + f(x_0) = \lim_{u \to 1} f(u) + f(x_0) = \cancel{f(1)}^{0} + f(x_0) = f(x_0)$. Let $\epsilon > 0$ be given, and take $n \in \mathbb{P}$ s.t. $n > \frac{f(2)}{\epsilon}$, $\delta := \min\{2^{1/n} - 1, 1 - 2^{-1/n}\}$ (>0). Then

$$1 - \delta < x < 1 + \delta \Rightarrow 2^{-1/n} < x < 2^{1/n} \underset{f \text{ str. incr.}}{\Longrightarrow} -\frac{1}{n} f(2) < f(x) < \frac{1}{n} f(2)$$

$\Rightarrow -\epsilon < f(x) < \epsilon$; so $\lim_{x \to 1} f(x) = 0 = f(1)$, done.

Thus it will suffice to prove the result assuming f continuous, and not identically zero. Then we have $f(\alpha) > 0$ for some $\alpha \in \mathbb{R}^+$, hence $f(\alpha^N) = N f(\alpha) > 1$ for some $N \in \mathbb{P}$ (Arch. property), while $f(1/\alpha) < 0$. By IVT, $\exists\, b \in \mathbb{R}^+$ s.t. $f(b) = 1$, and $F := f - \log_b$ is a continuous function satisfying $(*)$ which is 0 at b hence at any rational power of b. Given $x < y$, we know that between $\log_b x$ & $\log_b y$ there is some $q \in \mathbb{Q}$; and then $b^q \in (x, y)$ (otherwise strict monotonicity of $\log_b$ yields a contradiction). So if $F(c) \neq 0$ for some $c \in \mathbb{R}^+$, $\exists\, \delta > 0$ s.t. $x \in (c - \delta, c + \delta)$ $\Rightarrow |F(x) - F(c)| < F(c)/2 \Rightarrow F(x) \neq 0$. This is a contradiction since there is a "b^q" in $(c - \delta, c + \delta)$. So $F = 0$. $\square$

This is a stronger result than the book's, which assumes f is differentiable to get $f(wx) = f(w) + f(x) \xRightarrow{d/dx} w f'(wx) = f'(x)$
$\underset{x=1}{\Rightarrow} f'(w) = \frac{f'(1)}{w} \Rightarrow f(w) = \log_b(w)$ where $f'(1) = \frac{1}{\log(b)}$
(i.e. $b = \exp(1/f'(1))$).

Two quick applications

(1) $\int \frac{f'(x)}{f(x)} dx = \log f(x) + C$ if $f = u > 0$. If not, use instead $\mathcal{L}(x) := \log|x|$ on $\mathbb{R}\setminus\{0\}$ $\left(= \begin{cases} \log x, & x > 0 \\ \log(-x), & x < 0 \end{cases} \right)$.

Since $\mathcal{L}'(x) = \begin{cases} 1/x, & x > 0 \\ -1/-x, & x < 0 \end{cases} = \frac{1}{x}$, this makes the u-subst. above valid on any interval where $u \neq 0$.

Ex/ $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)} dx \underset{u=\cos(x)}{=} -\int \frac{du}{u} = -\log|u| + C = -\log|\cos x| + C$. //

(2) logarithmic differentiation: $g = \mathcal{L} \circ f \Rightarrow g' = (\mathcal{L}' \circ f) \cdot f' = \frac{f'}{f}$
$\Rightarrow f' = f g'$.

Ex/ $f(x) = x^2 (\cos x)(1+x^4)^{-7}$

$g(x) := (\mathcal{L} \circ f)(x) = 2\log|x| + \log|\cos(x)| - 7\log|1+x^4|$

$g'(x) = \frac{2}{x} - \frac{\sin(x)}{\cos(x)} - \frac{28x^3}{1+x^4}$

$f'(x) = fg' = \frac{2x\cos(x)}{(1+x^4)^7} - \frac{x^2 \sin x}{(1+x^4)^7} - \frac{28x^5 \cos x}{(1+x^4)^8}$. //

I should also point out that log gives a really nice approximation to $\pi(x) :=$ # of primes $\leq x$. In fact, $\pi(x) \sim \frac{x}{\log(x)}$ where "$\sim$" means "is asymptotic to", i.e. $\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1$. An even better approx. is $\int_2^x \frac{dt}{\log(t)}$.