Lecture 21 : The exponential function

<u>Definition</u>: Let $\exp : \mathbb{R} \to \mathbb{R}^+$ be the (strictly increasing, continuous) inverse function of $\log : \mathbb{R}^+ \to \mathbb{R}$ which we showed to exist in Lecture 20.

As we did with log, we take a deep dive into its properties:

(1) $(\exp \circ \log)(x) = x$ and $(\log \circ \exp)(x) = x$

(2) $\exp(0) = 1$, $\exp(1) = e$ (this was the definition of e)

(3) $\exp(a+b) = \exp(a)\exp(b)$

<u>Proof</u>: $\log(\exp(a)\exp(b)) = \log(\exp(a)) + \log(\exp(b))$
$= a+b = \log(\exp(a+b))$

Now apply exp to both sides. $\square$

(4) <u>Consequences of (3)</u>: (i) $\exp(-a) = \dfrac{1}{\exp(a)}$, (ii) $\exp(na) = (\exp(a))^n$, $n \in \mathbb{P}$

and (more generally) (iii) $\exp(ra) = (\exp(a))^r$ for any $r = \frac{p}{q} \in \mathbb{Q}$.

<u>Proof</u>: for (i), put $b = -a$ in (3); for (ii), use induction: $\exp((n+1)a)$ $\underset{(3)}{=} \exp(na)\exp(a) = \exp(a)^n \exp(a) = \exp(a)^{n+1}$. For (iii), write ($p \in \mathbb{Z}$, $q \in \mathbb{P}$)
$(\exp(a))^p = \exp(pa) = \exp(q \cdot \frac{p}{q} a) = (\exp(\frac{p}{q}a))^q \Rightarrow \exp(\frac{p}{q}a) = (\exp(a))^{p/q}$. $\square$

(5) exp is differentiable, with $\exp'(x) = \exp(x)$.

<u>Proof</u>: [Idea is that from $(\log \circ \exp)(x) = x$ the chain rule would give $1 = (\log \circ \exp)'(x) = \log'(\exp(x)) \cdot \exp'(x) = \dfrac{\exp'(x)}{\exp(x)}$, <u>if</u> we knew $\exp'(x)$ exists. Since we need to <u>prove</u> that, it will look different.]

Let

- $g(h) := \exp(h) - 1$, which is continuous with $g(0) = 0$, and is 0 nowhere else (by strict monotonicity); and
- $f(u) := \dfrac{u}{\log(1+u)}$, for which $\lim\limits_{u\to 0} f(u) = \dfrac{1}{\lim\limits_{u\to 0} \dfrac{\log(1+u) - \log(1)}{u}} = \dfrac{1}{\log'(1)} = 1.$

By the limit laws of compositions †

$$1 = \lim_{u \to g(0)} f(u) = \lim_{h\to 0} f(g(h)) = \lim_{h\to 0} \frac{g(h)}{\log(1+g(h))} \quad \leftarrow = \log(\exp(h)) = h$$

$$= \lim_{h \to 0} \frac{\exp(h) - 1}{h}$$

and so $(\exp'(x) =)$

$$\lim_{h\to 0} \frac{\exp(x+h) - \exp(x)}{h} \overset{(3)}{=} \exp(x) \cdot \lim_{h \to 0} \frac{\exp(h) - 1}{h} = \exp(x).$$

$\square$

(6) Immediate consequence of (5): $\exp(x)$ is strictly convex. (why?)

Taking $a = 1$ in (4)(iii), finally, gives

(7) $\exp(r) = (\exp(1))^r = e^r$ for any $r \in \mathbb{Q}$,

which suggests the following definition of real powers:

Definition: For any $x \in \mathbb{R}$, set $e^x := \exp(x)$, and more generally (for any $b \in \mathbb{R}^+$) $b^x := e^{x \log(b)} = \exp(x \log(b))$.
why?

How about properties of these?

(I) $b^0 = 1$

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† If g is continuous at p, and $\lim\limits_{y\to g(p)} f(y)$ exists, and $g(x) \neq g(p)$ for $x \in \mathcal{N}^*(p)$, then $\lim\limits_{x\to p} (f\circ g)(x) = \lim\limits_{y\to g(p)} f(y)$.

(II) $b^{x+y} = b^x b^y$ (follows from (3))

(III) $\log b^x = x \log(b)$ (follows from the Definition!)

(IV) $b^x = y \overset{(b \neq 1)}{\iff} x = \log_b y$ (how one usually thinks of $\log_b$!)

Proof: ($\Rightarrow$) Taking $\log_b$ of both sides, $\log_b y = \log_b b^x = \frac{\log b^x}{\log b}$

$= \frac{x \log(b)}{\log(b)} = x$. ($\Leftarrow$) left to you. $\square$

(V) $(ab)^x = e^{x \log(ab)} = e^{x \log a + x \log b} = e^{x\log(a)} e^{x \log(b)} = a^x b^x$

(VI) $a^{x+y} = e^{(x+y)\log a} = \cdots = a^x a^y$

(VII) $(a^x)^y = (a^y)^x = a^{xy}$

Proof: First, taking $b = e^x$ in $b^y = e^{y \log b}$, we get $(e^x)^y = e^{y \log(e^x)} = e^{xy}$. Now $a^{xy} = e^{xy \log(a)} = (e^{x \log(a)})^y = (a^x)^y$. $\square$

(VIII) $\frac{d}{dx} b^x = \frac{d}{dx} e^{x \log(b)} = \log(b) e^{x \log(b)} = b^x \log(b)$.

Ex/ What is $\frac{d}{dx} x^x$? Write $f(x) = x^x$, $g(x) = \log|f| = x \log|x|$

$\Rightarrow g'(x) = \log|x| + \frac{x}{x} = \log|x| + 1 \Rightarrow f'(x) (= fg') = x^x(\log|x| + 1)$. //

Ex/ $\int x^3 e^{-x^2} dx = -\frac{1}{2}(x^2+1) e^{-x^2} + C$ (Why?)

$\begin{bmatrix} u = -\frac{1}{2}x^2, & dv = -2x e^{-x^2} dx \\ du = -x\,dx, & v = e^{-x^2} \end{bmatrix}$

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Ex/ $\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$ (Why?)

$\begin{bmatrix} u = e^x & dv = \sin(x)\,dx \\ du = e^x dx & v = -\cos(x) \end{bmatrix}$ + 2nd $\int$-by-parts

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Ex/ $\int \frac{dx}{1+e^x} = x - \log(1+e^x) + C$ (Why?)

rewrite $\frac{1}{1+e^x} = 1 - \frac{e^x}{1+e^x}$

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Approximations to log (warmup to Taylor approximations next week)

For $x \in (-1,1)$,

$$\log\left(\frac{1}{1-x}\right) = -\log(1-x) = -\int_1^{1-x}\frac{dt}{t} \overset{t=1-u}{=} \int_0^x \frac{du}{1-u} = \int_0^x \frac{1-u^n+u^n}{1-u}\,du$$

$$\overset{\frac{1-u^n}{1-u} = 1+u+u^2+\dots+u^{n-1}}{=} \int_0^x \left(1+u+u^2+\dots+u^{n-1} + \frac{u^n}{1-u}\right)du = \underbrace{x+\frac{x^2}{2}+\frac{x^3}{3}+\dots+\frac{x^n}{n}}_{\text{polynomial}} + \underbrace{\int_0^x \frac{u^n}{1-u}du}_{\text{error}}.$$

$$=: P_n(x) + E_n(x)$$

polynomial error

Bound the error: for $x \in (0,1)$,

$$0 \le u \le x \Rightarrow 1 \ge 1-u \ge 1-x \Rightarrow 1 \le \frac{1}{1-u} \le \frac{1}{1-x} \Rightarrow u^n \le \frac{u^n}{1-u} \le \frac{u^n}{1-x}$$

$$\Rightarrow \frac{x^{n+1}}{n+1} = \int_0^x u^n\,du \le E_n(x) \le \frac{1}{1-x}\int_0^x u^n\,du = \frac{x^{n+1}}{(n+1)(1-x)}. \quad (*)$$

If $n=2m$, have

$$0 < -E_{2m}(-x) = \int_0^x \frac{u^n}{1+u}du \le \int_0^x \frac{u^n}{1-u}du = E_{2m}(x) \le \frac{x^{2m+1}}{(2m+1)(1-x)} \le \frac{x^{2m+1}}{2m+1}. \quad (**)$$

So

$$\boxed{\log\left(\frac{1+x}{1-x}\right) = \log\left(\frac{1}{1-x}\right) - \log\left(\frac{1}{1+x}\right) = P_{2m}(x) - P_{2m}(-x) + \underbrace{E_{2m}(x) - E_{2m}(-x)} = 2\left(x+\frac{x^3}{3}+\dots+\frac{x^{2m-1}}{2m-1}\right) + R_m(x)}$$

with (adding $(*)$ & $(**)$)

$$\boxed{\frac{x^{2m+1}}{2m+1} \le R_m(x) \le \frac{(2-x)}{(1-x)}\frac{x^{2m+1}}{2m+1}.}$$

You'll use this to estimate $\log(2)$ in the HW.