Lecture 22: Derivatives of inverses

<u>Theorem</u>: Suppose f is strictly monotonic and continuous on an interval I. Then it has a unique (strictly monotonic and continuous) inverse function g. [We already know this.] Moreover, if f is differentiable at x, then g is differentiable at $f(x)$, with $g'(f(x)) = \dfrac{1}{f'(x)}$.

Remark: This is often written $(f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$, by writing $g = f^{-1}$ and substituting $f^{-1}(y)$ for x.

This looks really nontrivial. Or (thinking $y = f(x)$, $x = f^{-1}(y)$) you can write it in the much more trivial-looking form $dx/dy = \dfrac{1}{dy/dx}$.

<u>Proof</u>: We argue exactly as in the proof for exp: write

$$G(h) := g(f(x)+h) - g(f(x)) \quad \text{and} \quad F(u) := \frac{u}{f(x+u)-f(x)}.$$

Then $\lim\limits_{u\to 0} F(u) = \dfrac{1}{\lim\limits_{u\to 0} \dfrac{f(x+u)-f(x)}{u}} = \dfrac{1}{f'(x)}$, while (by continuity & strict monotonicity of g) G is continuous at 0 with $G(0) = 0$ and $G(h) \neq 0$ for $h \in N^*(0)$. By the limit laws for compositions, $\lim\limits_{h\to 0} F(G(h)) = \lim\limits_{u\to G(0)(=0)} F(u) = \dfrac{1}{f'(x)}$ on the one hand; while on the other

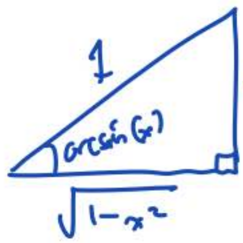
$$\lim_{h\to 0} F(G(h)) = \lim_{h\to 0} \frac{G(h)}{f(x+G(h)) - f(x)} = \lim_{h\to 0} \frac{g(f(x)+h) - g(f(x))}{h} = g'(f(x)).$$

(Here we used that $f(x+g(f(x)+h)-g(f(x)))-f(x) =$
$f(\cancel{x}+g(f(x)+h)-\cancel{x})-f(x) = f(g(f(x)+h))-f(x) = f(x)+h-f(x) = h.$) $\square$

range of sin on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Ex/ Define $\arcsin(x) := \sin^{-1}(x)$, on the interval $[-1, 1]$.

On $(-1, 1)$,
$$\arcsin'(x) = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))}$$



$$= \frac{1}{\sqrt{1-x^2}}.$$ //

range of tan on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

Ex/ Define $\arctan(x) := \tan^{-1}(x)$, on $\mathbb{R}$. We have

$$\arctan'(x) = \frac{1}{\tan'(\arctan(x))} = \frac{1}{(\sec(\arctan(x)))^2} = [\cos(\arctan(x))]^2$$

$$= \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}.$$ //

Ex/
$$\int \frac{dx}{\sqrt{1-2x-x^2}} = \int \frac{dx}{\sqrt{2-(x+1)^2}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{1-\left(\frac{x+1}{\sqrt{2}}\right)^2}}$$
$$= \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + C = \arcsin\left(\frac{x+1}{\sqrt{2}}\right) + C$$

$u = \frac{x+1}{\sqrt{2}}$, $du = \frac{dx}{\sqrt{2}}$

//

Hyperbolic functions Form

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{We define } \tanh = \frac{\sinh}{\cosh}$$

and so forth in analogy to trigonometric functions. (In fact, it's more than an analogy, as we'll see when we get to complex #s.)

Why "hyperbolic"? Because they satisfy the equation $x^2 - y^2 = 1$ of a hyperbola: $\cosh^2 x - \sinh^2 x = \dfrac{e^{2x} + 2 + e^{-2x}}{4} - \dfrac{e^{2x} - 2 + e^{-2x}}{4} = 1.$

We also have

- $\sinh(-x) = \dfrac{e^{-x} - e^{x}}{2} = -\sinh(x)$, $\cosh(-x) = \cosh(x)$
- $\frac{d}{dx} e^{-x} = -e^{-x} \Rightarrow \sinh'(x) = \cosh(x)$, $\cosh'(x) = \sinh(x)$; and
$$\tanh'(x) = \frac{\cosh(x)^2 - \sinh(x)^2}{\cosh(x)^2} = \frac{1}{\cosh(x)^2} =: \operatorname{sech}^2(x)$$
- $$\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \frac{(e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})}{4}$$
$$= \frac{e^{x+y} - \cancel{e^{-x+y}} + \cancel{e^{x-y}} - e^{-x-y} + e^{x+y} + \cancel{e^{-x+y}} - \cancel{e^{x-y}} - e^{-x-y}}{4} = \sinh(x+y),$$

and many other identities analogous to the trigonometric ones. We may even consider (on $\mathbb{R}$)

$$(\tanh^{-1})'(x) = \frac{1}{\tanh'(\tanh^{-1}(x))} = [\cosh(\underbrace{\tanh^{-1}(x)}_{=: y})]^2$$

$$\Rightarrow \tanh(y) = x$$
$$\Rightarrow x^2 = \frac{\sinh^2 y}{\cosh^2 y} = 1 - \frac{1}{\cosh^2 y}$$
$$\Rightarrow \cosh^2(y) = \frac{1}{1-x^2}$$

$$= \frac{1}{1-x^2}$$

which is very interesting given that

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{1/2}{1-x} + \frac{1/2}{1+x} \Rightarrow$$

$$\frac{d}{dx}\left(\frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)\right) = \frac{1}{1-x^2}.$$

Since $\tanh^{-1}(0) = 0 = \frac{1}{2}\log\left(\frac{1+0}{1-0}\right)$, it implies that $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$ on $(-1,1)$.
Can you prove this more directly?

Long division for polynomials

Proposition: Given polynomials $F(x)$ & $G(x)$ of degree ≥ 1, there exist **unique** polynomials $q(x)$ and $r(x)$ such that

$$(*) \qquad F = qG + r \quad \text{and} \quad \deg(r) < \deg(G).$$

Ex/ $F = x^3 + x + 3$
$G = x + 1$

$$\begin{array}{r l}
 & \phantom{x^3+{}} x^2 - x + 2 \\
x+1 & \overline{)\, x^3 + 0x^2 + x + 3} \\
 & -(x^3 + x^2) \\
\hline
 & \qquad -x^2 + x + 3 \\
 & \qquad -(-x^2 - x) \\
\hline
 & \qquad\qquad 2x + 3 \\
 & \qquad\qquad -(2x + 2) \\
\hline
 & \qquad\qquad\qquad 1
\end{array}$$

$q = x^2 - x + 2$ $\Longleftarrow$
$r = 1$

(Of course "q" stands for quotient, and "r" remainder.) //

Proof: Let $\mathcal{S} := \{F - pG \mid p \text{ polynomial}\}$, where $G = ax^n + \text{lower-degree terms}$. Let $r = bx^m + \text{lower deg. terms}$ be an element of $\mathcal{S}$ of lowest degree. If $m \geq n$, then $r - \frac{b}{a}x^{m-n}G$ is an element of $\mathcal{S}$ of degree less than m, a contradiction. This proves existence.

For uniqueness, suppose $F = qG + r = QG + R$ both satisfy $(*)$. If $q \neq Q$ then the left-hand side of

$$(q - Q)\,G = R - r$$

has strictly greater degree than $R - r$, which is absurd. So $q = Q$, hence $R = r$. $\square$