Lecture 23: Partial fractions

The basic result that makes integration by partial fractions work is the following

<u>Theorem</u>: Let $\mathcal{F}$ and $\mathcal{G}$ be polynomials. Then

(a) $\mathcal{G} = G_1^{k_1} G_2^{k_2} \cdots G_n^{k_n}$ with each G_i linear $(ax+b)$ or irreducible quadratic (ax^2+bx+c), and

(b) $\dfrac{\mathcal{F}}{\mathcal{G}} = Q + \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{P_{ij}}{G_i^j}$ where Q and P_{ij} are (the unique) polynomials with $\deg(P_{ij}) < \deg(G_i)$.

Part (a) is a version of the Fundamental Theorem of Algebra, which I'll discuss when we get to complex numbers. The main point today is part (b), which reduces the computation of any $\int \frac{\mathcal{F}}{\mathcal{G}}\,dx$ to computation of

(i) $\displaystyle\int \frac{c\,dx}{(ax+b)^k}$ — use power rule or $\log|\cdot|$

and

(ii) $\displaystyle\int \frac{(\alpha x+\beta)\,dx}{(ax^2+bx+c)^k}$ — use a combination of u-substitution and arctan + completing the square.

you can do <u>all</u> of these integrals!

Ex/ $\displaystyle\int \frac{x+1}{x^2+x+1}\,dx = \int \frac{\frac{1}{2}\,du}{u} + \int \frac{\frac{1}{2}\,dx}{x^2+x+1}$

$(x+\frac{1}{2})^2 + \frac{3}{4}$

[example of "most general" type of (ii)]

$\left(\begin{array}{c} u = x^2+x+1 \\ \frac{1}{2}du = (x+\frac{1}{2})\,dx \end{array}\right)$

$$= \frac{1}{2}\log|u| + \frac{2}{3}\int \frac{dx}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}$$

$\left(\begin{array}{c} v = \frac{2x+1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}}dv = \frac{2}{3}dx \end{array}\right)$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\int \frac{dv}{1+v^2}$$

$$= \frac{1}{2}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C. \quad //$$

To prove (the existence part of) (b), recall from last time that given polynomials $g(x)$ & $f(x)$, with g not identically 0, we have (unique) "quotient" and "remainder" polynomials q & r, with

$$(*)\quad \boxed{f = qg + r \quad \text{and} \quad \deg(r) < \deg(g).}$$

Note: (a) If g is a constant (degree 0), then $q = \frac{f}{g}$ and $r = 0$. We consider the degree of "0" to be -1. (This is just a convention so that $(*)$ holds for $\deg(g) = 0$ too.)

(b) Automatically, $\deg(q) = \deg(f) - \deg(g)$.

So given $\frac{\mathcal{F}}{\mathcal{G}}$, we can write $\mathcal{F} = Q\mathcal{G} + R$ to get $Q + \frac{R}{\mathcal{G}}$ with $\deg(R) < \deg(\mathcal{G})$; that is, we may assume that $\deg(\mathcal{F}) < \deg(\mathcal{G})$. Moreover, it will suffice in that case to show that:

(I) if $\mathcal{G} = \mathcal{G}_1 \mathcal{G}_2$ where $\mathcal{G}_1$ & $\mathcal{G}_2$ share no common factor, then

$$(**) \qquad \frac{\mathcal{F}}{\mathcal{G}} = \frac{F_1}{\mathcal{G}_1} + \frac{F_2}{\mathcal{G}_2} \quad \text{with } \deg F_i < \deg \mathcal{G}_i .$$

(II) if $\mathcal{G} = G^k$ with G irreducible, then

$$\binom{**}{*} \qquad \frac{\mathcal{F}}{\mathcal{G}} = \frac{f_1}{G} + \frac{f_2}{G^2} + \cdots + \frac{f_k}{G^k} \quad \text{with } \deg f_i < \deg G .$$

These are enough because now if $\mathcal{G} = \underbrace{G_1^{k_1}}_{\mathcal{G}_1} \underbrace{(G_2^{k_2} \cdots G_n^{k_n})}_{\mathcal{G}_2}$

we can write $\dfrac{\mathcal{F}}{\mathcal{G}} = \dfrac{F_1}{G_1^{k_1}} + \dfrac{F_2}{\mathcal{G}_2}$ then apply (II) to the first term and (I) (again) to the second, and repeat.

It's easy to prove (II): writing

$$\mathcal{F} =: \mathcal{F}_k = \mathcal{F}_{k-1} G + f_k \quad \text{with } \begin{cases} \deg f_k < \deg G \\ \deg \mathcal{F}_{k-1} = \deg \mathcal{F}_k - \deg G < \deg G^{k-1} \end{cases}$$

we have

$$\frac{\mathcal{F}}{G^k} = \frac{\mathcal{F}_{k-1} G + f_k}{G^k} = \frac{\mathcal{F}_{k-1}}{G^{k-1}} + \frac{f_k}{G^k} .$$

Now repeat this for $\mathcal{F}_{k-1}/G^{k-1}$, eventually arriving at $\binom{**}{*}$.

So the most important part is really (I). This will follow from the

<u>Lemma</u>: If F and G are polynomials (of deg ≥ 1) with no common factor, then there exist polynomials A and B with $\deg A < \deg F$, $\deg B < \deg G$, s.t. $AG + BF = 1$.

Why? Well, if $AG_2 + BG_1 = 1$, then

$$\frac{A}{G_1} + \frac{B}{G_2} = \frac{AG_2 + BG_1}{G} = \frac{1}{G}$$

$$\underset{\times \sigma}{\Rightarrow} \quad \frac{\sigma}{G} = \frac{A\sigma}{G_1} + \frac{B\sigma}{G_2} \underset{\substack{\uparrow \\ \text{long division} \\ (\deg R_i < \deg G_i)}}{=} \frac{R_1}{G_1} + Q_1 + \frac{R_2}{G_2} + Q_2 \, ,$$

at which point it is clear that $Q_1 + Q_2 = 0$. †

<u>Proof of Lemma</u>: Write $F_0 = F$, $F_1 = G$, assume $\deg F_1 \le \deg F_0$.

Now apply long division repeatedly to get

$$F_0 = F_1 q_1 + F_2 \, , \quad \deg F_2 < \deg F_1$$

$$F_1 = F_2 q_2 + F_3 \, , \quad \deg F_3 < \deg F_2$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$F_{n-2} = F_{n-1} q_{n-1} + F_n$$

$$F_{n-1} = F_n q_n + 0$$

this can't go on forever — eventually we reach "-1", i.e. $F_{n+1} = 0$

Plugging these in "backwards" gives

$$F_{n-2} = F_{n-1} q_{n-1} + F_n = F_n q_n q_{n-1} + F_n = F_n (q_n q_{n-1} + 1)$$

$$F_{n-3} = F_{n-2} q_{n-2} + F_{n-1} = F_n (q_n q_{n-1} + 1) q_{n-2} + F_n q_n$$

$$\vdots$$

and as you can see, every F_k is $F_n \times (\text{something})$. That is, at the end of the day F_n divides F_0 $(= F)$ and F_1 $(= G)$.

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† $$Q_1 + Q_2 = \frac{\sigma}{G} - \left(\frac{R_1}{G_1} + \frac{R_2}{G_2} \right) = \frac{\sigma - (R_1 G_2 + R_2 G_1)}{G}$$

with $\deg \sigma$, $\deg R_1 G_2$, $\deg R_2 G_1$ all $< \deg G$. This is absurd unless the numerator (hence $Q_1 + Q_2$) is 0.

But F & G have no common factor. So F_n is a <u>constant</u> c.

Finally, write $\mathcal{A}(k)$ for the statement that $F_k = A_k G + B_k F$. Clearly $\mathcal{A}(2)$ is true, b/c

$$F_2 = F_0 - F_1 q_1 = 1 \cdot F + (-q_1) G$$

If $\mathcal{A}(m)$ holds for $m \le k$, then

$$\begin{aligned} F_{k+1} = F_{k-1} - F_k q_k &= A_{k-1} G + B_{k-1} F - (A_k G + B_k F) q_k \\ &= (A_{k-1} - A_k q_k) G + (B_{k-1} - B_k q_k) F \end{aligned}$$

$\Rightarrow \mathcal{A}(k+1)$. So $\mathcal{A}(n)$ holds — $c = A_n G + B_n F$ — and dividing by c gives

$$1 = \tilde{A} G + \tilde{B} F.$$

Finally, if either $\deg \tilde{A} \ge \deg F$ or $\deg \tilde{B} \ge \deg G$, write

$$\tilde{A} = q_A F + \overset{\deg < \deg F}{r_A}, \quad \tilde{B} = q_B G + \overset{\deg < \deg G}{r_B},$$

with at most one of $q_A, q_B \ne 0$. We then have

$$1 = \underbrace{(q_A + q_B) F G}_{\substack{\deg \ge \deg FG \\ (\text{if not } 0)}} + \underbrace{r_A G + r_B F}_{\deg < \deg FG},$$

which is absurd, unless $q_A + q_B = 0$. So taking $A := r_A$ and $B := r_B$, we have $AG + BF = 1$, $\deg A < \deg F$, and $\deg B < \deg G$. $\square$