# Lecture 24: Taylor polynomials

Assume $f$ is $n$-times differentiable at $a \in \mathrm{Dom}(f)$.

This means that $f'(a), f''(a), \ldots, f^{(n)}(a)$ all exist.

<u>Definition</u>: The <u>$n^{\text{th}}$ Taylor polynomial of $f$ at $a$</u> is

$$\underbrace{(T_{n,a}f)}_{\text{or } T_n f, \text{ or just } T_n}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

These will help us to estimate values of transcendental functions and to evaluate indeterminate forms.

<u>Proposition 1</u>: $T_n f$ is the unique polynomial $P$ of degree $\leq n$ with $P^{(j)}(a) = f^{(j)}(a)$ for all $j = 0, 1, \ldots, n$.

<u>Proof</u>: Given $P = \sum_{k=0}^{n} c_k (x-a)^k$, differentiating $j$ times gives

$$P^{(j)} = \sum_{k=j}^{n} c_k \cdot \underbrace{k(k-1)\cdots(k-j+1)}_{k!/(k-j)!} \cdot (x-a)^{k-j}$$

so that $f^{(j)}(a) = P^{(j)}(a) = c_j \cdot j! \iff c_j = \dfrac{f^{(j)}(a)}{j!}$. $\square$

Ex/ $(T_n \exp)(x) = \sum_{k=0}^{n} x^k/k!$

$$(T_{2n+1} \sin)(x) = (T_{2n+2} \sin)(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$(T_{2n} \cos)(x) = (T_{2n+1} \cos)(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

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Ex/ $f(x) = (1+x)^r$, $r \in \mathbb{R}$.

$f^{(k)}(x) = r(r-1)\cdots(r-k+1)(1+x)^r \Rightarrow f^{(k)}(0) = r(r-1)\cdots(r-k+1)$

$\Rightarrow (T_n f)(x) = \sum_{k=0}^{n} \binom{r}{k} x^k$, where $\binom{r}{k} := \frac{r(r-1)\cdots(r-k+1)}{k!}$.

e.g. if $r = \frac{1}{2}$, $T_4 = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4$. //

To get more examples, we need ...

Properties: (1) $T_n(\alpha f + \beta g) = \alpha T_n f + \beta T_n g$

(2) $(T_{n+1} f)' = T_n(f')$

(3) $(T_n \int_a f(t)\,dt)(x) = \int_a^x (T_{n-1} f)(t)\,dt$

(4) $(T_{n,a}(f \circ h)) = (T_{n,ca} f) \circ h$ if $h(x) = cx$

Proofs: In all cases, just compute values at $a$ of $0^{\text{th}}$ thru $n^{\text{th}}$ derivatives (since all are polynomials of degree $n$ hence determined by these values). e.g., for (3) $T_{n-1} f$ has same derivatives [at $a$] as $f$ so $\int_a^x (T_{n-1} f)(t)\,dt$ (being its primitive) has same derivatives as $\int_a^x f(t)\,dt$, hence LHS (3). For (4), both sides share derivatives at $a$ with $(f \circ h)^{(k)}(a) = c^k f^{(k)}(ca)$. $\square$

Ex/ $(T_{2n+1} \sinh)(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!}$, $(T_{2n} \cosh)(x) = \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$

Use (1) + (4) with $c = -1$, $a = 0$ $\left(\Rightarrow (T_n e^{-x}) = \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!}\right)$. //

Proposition 2: Suppose $P$ is a polynomial of degree $n$ with $f(x) = P(x) + (x-a)^n g(x)$, where $f$ & $g$ are $n$-times differentiable at $a$ and $g(a) = 0$. Then $P = T_n f$.

Proof: Put $h(x) = (x-a)^n g(x)$. Since $h^{(k)}(a) = 0$ for $0 \le k \le n$, $f^{(k)}(a) = P^{(k)}(a)$, done by Prop. 1. $\square$

Property (5): In the situation of Prop. 2 with $a=0$, the $ln^{\text{th}}$ Taylor poly. of $f(x^l)$ is $(T_n f)(x^l)$.

Proof: $f(x) = (T_n f)(x) + x^n g(x)$

$$\Rightarrow f(x^l) = (T_n f)(x^l) + x^{ln} g(x^l) \quad \text{is still 0 at 0}$$

$$\underset{\text{Prop. 2}}{\Rightarrow} (T_{ln} f)(x^l) = T_n(f(x^l)). \qquad \square$$

Ex/ $\underbrace{\frac{1}{1-x}}_{f} = \underbrace{1+x+\cdots+x^n}_{P} + x^n \cdot \underbrace{\frac{x}{1-x}}_{g} \underset{\text{Prop. 2}}{\Longrightarrow} T_n f = \sum_{k=0}^{n} x^k$

- $\underset{(2)}{\Longrightarrow} T_n \frac{1}{(1-x)^2} = T_n\left(\frac{d}{dx}\frac{1}{1-x}\right) = \frac{d}{dx} T_{n+1} f = \sum_{k=1}^{n+1} k x^{k-1} = \sum_{k=0}^{n} (k+1) x^k.$
- $\underset{(3)}{\Longrightarrow} -\log(1-x)$ has $T_n = \sum_{k=1}^{n} \frac{x^k}{k}$
- $\underset{(5) + (4) \text{ with } c=-1}{\Longrightarrow} \frac{1}{1+x^2}$ has $T_{2n} = (T_n f)(-x^2) = \sum_{k=0}^{n} (-1)^k x^{2k}$
- $\underset{(3)}{\Longrightarrow} \arctan(x)$ has $T_{2n+1} = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}$. //

Definition: The $n^{\text{th}}$ error or remainder is $E_n(x) := f(x) - T_n(x)$.

Theorem: (a) If $f^{(n+1)}$ is continuous, then $E_n(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$.

(b) If $f^{(n)}$ is continuous and $f^{(n+1)}$ exists, then $E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some $c \in (a, x)$.

Proof: (a) Induce on $n$.

$\underline{n=0}$: $E_0(x) = f(x) - T_0(x) = f(x) - f(a) = \int_a^x f'(t)\,dt$

$\underline{n \Rightarrow n+1}$: $E_{n+1}(x) = f(x) - T_{n+1}(x) = \underbrace{f(x) - T_n(x)}_{E_n(x)} - \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$

$$= \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt - \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1}$$

$$= \int_a^x \frac{f^{(n+2)}(t)}{(n+1)!}(x-t)^{n+1}\,dt.$$

$\int$ by parts with $u = \frac{(x-t)^{n+1}}{n+1}$, $dv = \frac{f^{(n+2)}(t)}{n!}\,dt$

$du = -(x-t)^n$, $v = \frac{f^{(n+1)}(t)}{n!}$

(b) Can deduce from (a) via weighted MVT if the stronger hypothesis of (a) is assumed.† To avoid this, work instead

$$g(t) := f(x) - \sum_{k=0}^n \frac{f^{(k)}(t)}{k!}(x-t)^k - E_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}} \quad (\Rightarrow g(x) = 0 = g(a))$$

$$g'(t) = 0 - \sum_{k=0}^n \frac{f^{(k+1)}(t)}{k!}(x-t)^k + \underbrace{\sum_{k=1}^n \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1}}_{= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k} + E_n(x)(n+1)\frac{(x-t)^n}{(x-a)^{n+1}}$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + E_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}.$$

MVT $\Rightarrow \exists\, c \in (a,x)$ s.t. $g'(c) = 0$

$$\Rightarrow E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \qquad \square$$

We'll put these forms of the remainder to good use next week.

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† $\exists\, c$ s.t. $\frac{f^{(n+1)}(c)}{n!} = \frac{\int_a^x \frac{f^{(n+1)}(t)}{n!}\cdot(x-t)^n\,dt}{\int_a^x (x-t)^n\,dt} = \frac{E_n(x)}{\left(\frac{(x-a)^{n+1}}{n+1}\right)}$

$\Rightarrow E_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$