Lecture 25 : Bounding the remainder

Given a function f with continuous $(n+1)^{\underline{st}}$ derivative in a neighborhood $\mathcal{N}(a) = (a-c, a+c)$ of a, we derived the formula

$$E_n(x) := f(x) - T_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$$

for the error in the $n^{\underline{th}}$ Taylor approximation (polynomial) at a.

This leads at once to the

Theorem 1: $$|E_n(x)| \le \frac{\sup_{t \in [a,x] \text{ or } [x,a]} |f^{(n+1)}(t)|}{(n+1)!} |x-a|^{n+1}.$$

Proof: In general, we can bound integrals of continuous functions by reasoning that $-|F(t)| \le F(t) \le |F(t)| \Rightarrow -\int_a^b |F| \, dt \le \int_a^b F \, dt \le \int_a^b |F| \, dt$

$$\Rightarrow \left| \int_a^b F(t) \, dt \right| \le \int_a^b |F(t)| \, dt.$$ Writing $\mu := \sup_{t \in [a,x]} |f^{(n+1)}(t)|$, we have

$$|E_n(x)| \le \int_a^x \frac{|f^{(n+1)}(t)|}{n!} |x-t|^n \, dt \le \int_a^x \frac{\mu}{n!} |x-t|^n \, dt = \frac{\mu}{(n+1)!} |x-a|^{n+1}.$$ $\square$

Corollary: Writing $M := \sup_{x \in \mathcal{N}(a)} |f^{(n+1)}(x)|$, the error function $|E_n(x)| \le \frac{M}{(n+1)!} |x-a|^{n+1}$ on $\mathcal{N}(a)$. In particular, $\lim_{x \to a} \frac{E_n(x)}{(x-a)^n} = 0$.

Definition: If $g(x) \ne 0$ for $x \in \mathcal{N}^*(a)$ and $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$, then we write

$$f(x) = o(g(x)).$$

So in the Corollary we have $\underline{E_n(x) = o((x-a)^n)}$.

Why might we care about bounding the error?

(1) So we can decide how closely $T_n(x)$ approximates $f(x)$ at particular values of x. For instance, with $f(x) = e^x$ & $a = 0$,

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| = \left| E_n(1) \right| \le \frac{\sup_{t \in [0,1]} e^t}{(n+1)!} |1-0|^{n+1} = \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.$$

[To see that $e < 3$, notice that $\sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n!} <$
$\left(1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right) - \left(\frac{1}{2^2} - \frac{1}{2 \cdot 3}\right) = \left(3 - \frac{1}{2^n}\right) - \frac{1}{12}$, and
$e = \sum_{k=0}^{n} \frac{1}{k!} + E_n(1) < 3 - \frac{1}{2^n} - \frac{1}{12} + \frac{e}{(n+1)!} < 3$ by choosing n large enough that $\frac{e}{(n+1)!} < \frac{1}{12}$ — which would happen regardless of how big e was. Alternatively, you could use step functions to show that $\int_1^3 \frac{dx}{x} > 1$.]

This allows us to show that, for example, $\sum_{k=0}^{10} \frac{1}{k!}$ gives at least 6 decimal places of accuracy, since $\frac{3}{11!} \sim 0.00000008$.

(2) So we can show that certain numbers are irrational.

Theorem 2: Given sequences $a_n, b_n \in \mathbb{Z}$, and $r \in \mathbb{R}$, suppose that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ s.t.

$$n \ge N \implies 0 < \left| r - \frac{b_n}{a_n} \right| < \frac{\epsilon}{a_n}.$$

Then $r \notin \mathbb{Q}$.

Proof: If $r = \frac{p}{q}$ $(p, q \in \mathbb{Z})$, then taking $\epsilon = \frac{1}{q}$ gives

$$0 < \left| \frac{p}{q} - \frac{b_n}{a_n} \right| < \frac{1}{q a_n} \implies 0 < \left| \underbrace{p a_n - b_n q}_{\in \mathbb{Z}} \right| < 1 \quad \text{※} \qquad \square$$

Ex/ $r = e$, $a_n = n!$, $b_n = \sum_{k=0}^{n} \frac{n!}{k!}$. Put $f(x) = e^x$.

Of course $b_n/a_n = \sum_{k=0}^{n} \frac{1}{k!} = T_n(1)$. So the bound on $E_n(1)$ above gives

$$\left| e - \frac{b_n}{a_n} \right| = |E_n(1)| < \frac{3}{(n+1)!} = \frac{3/(n+1)}{n!} = \frac{3/(n+1)}{a_n}.$$

Taking $n \geq N > \frac{3}{\epsilon}$ makes $\frac{3}{n+1} < \epsilon$. Moreover, we know $E_n(1) = \int_0^1 \frac{f^{(n+1)}(t)}{n!}(1-t)^n\,dt = \frac{1}{n!}\int_0^1 e^t (1-t)^n\,dt > 0$.

So the hypotheses of Theorem 2 hold, and thus $e \notin \mathbb{Q}$. //

③ <u>In order to compute indeterminate forms</u> (i.e. limits of type $\frac{0}{0}$, 1^∞, etc.)

We shall make use of the following rules for computing with little-o notation: $o(g(x)) \pm o(g(x)) = o(g(x))$; $o(cg(x)) = o(g(x))$ if $c \neq 0$; $f(x)\cdot o(g(x)) = o(f(x)g(x))$; $o(o(g(x))) = o(g(x))$; and if $\lim_{x\to a} g(x) = 0$, $\frac{1}{1+g(x)} = 1 - g(x) + g(x)\frac{g(x)}{1+g(x)} = 1 - g(x) + o(g(x))$.

Ex/ Compute $L = \lim_{x \to 0} \frac{\alpha^x - \beta^x}{x}$. [In the examples, $a = 0$.]

By the Corollary, taking $f(t) = e^t$ we have $E_1(t) = o(t)$.

So $e^t = 1 + t + o(t)$, and substitution yields

$$\alpha^x - \beta^x = e^{x\log\alpha} - e^{x\log\beta} = \cancel{1} + x\log\alpha + \overset{o(x)}{\cancel{o(x\log\alpha)}} - \cancel{1} - x\log\beta - \overset{o(x)}{\cancel{o(x\log\beta)}}$$
$$= \left(\log\frac{\alpha}{\beta}\right)x + o(x)$$

$$\Rightarrow \frac{\alpha^x - \beta^x}{x} = \log\left(\frac{\alpha}{\beta}\right) + o(1) \quad \Rightarrow \quad L = \log\left(\frac{\alpha}{\beta}\right).$$

means $\to 0$ as $x \to 0$

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Ex/ Compute $L = \lim_{x \to 0} \frac{\log(1+bx)}{x}$ and $\mathcal{L} = \lim_{x \to 0} (1+bx)^{1/x}$, $b \in \mathbb{R}$.

Taking $f(x) = \log(1+bx)$ we have $f'(x) = \frac{b}{1+bx} \Rightarrow (T_1 f)(x) = bx$.

So $\log(1+bx) = bx + o(x) \Rightarrow \frac{\log(1+bx)}{x} = b + o(1) \Rightarrow L = b$.

Hence also $\mathcal{L} = \lim_{x \to 0} e^{1/x \log(1+bx)} = e^{\lim_{x \to 0} \frac{1}{x} \log(1+bx)} = e^b$ by continuity of exp.

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Ex/ Find $(T_3 \tan)(x)$.

We have $\sin(x) = x - \frac{x^3}{6} + O(x^4)$, $\cos(x) = 1 - \frac{x^2}{2} + O(x^3)$

$$\Rightarrow \frac{1}{\cos(x)} = \frac{1}{1 - \left(\frac{x^2}{2} + O(x^3)\right)} = 1 + \frac{x^2}{2} + O(x^2)$$

$$\Rightarrow \tan(x) = \frac{\sin(x)}{\cos(x)} = \left(x - \frac{x^3}{6} + O(x^4)\right)\left(1 + \frac{x^2}{2} + O(x^2)\right)$$

$$= x + \frac{x^3}{3} + O(x^3).$$

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The Taylor polynomials for $\tan(x)$ more generally are very mysterious. For instance, T_7 is $x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!}$. The numbers appearing in the numerators are of course the derivatives of tan at 0, the even ones being zero. The odd ones are given by a beautiful formula involving Bernoulli numbers:

$$\tan^{(2m-1)}(0) = (-1)^{m-1} \frac{(4^{2m} - 4^m)}{2m} B_{2m}.$$