Lecture 26 : L'Hôpital's rule

Having used Taylor polynomials to evaluate some indeterminate forms, we now discuss their main competitor. (Recall that "indeterminate forms" refers to limits of type $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, 0^0, ∞^0, and 1^∞.) Should probably be called "Bernoulli's rule".

Version 1: Given f, g, f', g' defined on (a,b), with $g' \neq 0$ there, and $\lim_{x\to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$. Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$, if the right-hand side exists. [If $b<a$, use (b,a) & $x \to a^-$.]

Ex/
$$\lim_{x\to 0} \frac{x - \tan x}{x - \sin x} \overset{L}{=} \underbrace{\lim_{x\to 0} \frac{1-\sec^2 x}{1-\cos x}}_{\text{Still } \frac{0}{0} \text{ : important to check this, otherwise cannot use "L" a 2nd time}} \overset{L}{=} \lim_{x\to 0} \frac{-2\sec^2 x \tan x}{\sin x}$$
$$= \lim_{x\to 0} \frac{-2}{\cos^3 x} = \frac{-2}{\cos^3(0)} = -2. \quad //$$

Ex/
$$\lim_{x\to 0} \frac{\sqrt{x}}{\underbrace{1-e^{2\sqrt{x}}}_{}} = \lim_{y\to 0} \frac{y}{1-e^{2y}} \overset{L}{=} \lim_{y\to 0}\frac{1}{-2e^{2y}} = \frac{1}{-2e^0} = \frac{-1}{2}.$$

think of as $f(g(x))$, $g(x) = \sqrt{x}$ cts. at 0 & nonzero away from 0 $\Big] \implies$ rules for limit of a composition apply

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[**Problem**: Use L'Hôpital or Taylor polynomials to evaluate
$$\lim_{x\to 0} \frac{\sqrt[3]{1+x} - 1 - \frac{1}{3}x}{x^2}.$$

Definition:
$$\lim_{\substack{x\to\infty \\ -\infty}} f(x) = L \iff \left\{ \begin{array}{l} \forall \epsilon > 0 \ \exists M \overset{<}{>} 0 \text{ s.t.} \\ x \underset{<}{>} M \implies |f(x) - L| < \epsilon. \end{array} \right.$$

Ex/ $\lim_{x\to\infty} e^{-x} = 0$: given ϵ, take $M = -\log\epsilon$.

Then $x > M \Rightarrow e^{x} > e^{-\log\epsilon} = \frac{1}{\epsilon}$ (since e^x is strictly increasing)

$\Rightarrow 0 < e^{-x} < \epsilon \Rightarrow |e^{-x} - 0| < \epsilon$. //

Version 2: Given f, g, f', g' defined for $x > A$, with $g' \neq 0$ there, and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x)$. Then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$.

This is especially useful in conjunction with the

$\overset{I}{\underline{\text{Inversion trick}}}$: $\lim_{\substack{x\to\infty\\ 0^+}} f(x) = L \iff \lim_{\substack{x\to 0^+\\ \infty}} f(\frac{1}{x}) = L$

[Problem: try $\lim_{x\to 0^+}\frac{e^{-1/x}}{x}$, $\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$.

Definition: $\lim_{x\to a} f(x) = \infty \iff \begin{cases} \forall M > 0 \ \exists \delta > 0 \text{ s.t.} \\ 0 < |x-a| < \delta \Rightarrow f(x) > M. \end{cases}$

[Problem: formulate a definition of $\lim_{x\to\infty} f(x) = -\infty$.

Version 3: Same as V.1 but with $\lim_{x\to a^+} f(x) = \infty = \lim_{x\to a^+} g(x)$.

Ex/ $\lim_{x\to\infty}\frac{(\log x)^b}{x^a} = 0 = \lim_{x\to\infty}\frac{x^b}{e^{ax}}$ for any $a, b \in \mathbb{R}^+$: by continuity of x^b, its enough to show their $1/b$ th powers $\left(\frac{\log x}{x^c} \text{ and } \frac{x}{e^{cx}} \text{ where } c = \frac{a}{b}\right) \to 0$.

This is easy: $\lim_{x\to\infty}\frac{\log x}{x^c} \overset{L}{=} \lim_{x\to\infty}\frac{1/x}{cx^{c-1}} = \lim_{x\to\infty}\frac{1}{cx^c} = 0$,

$$\lim_{x\to\infty}\frac{x}{e^{cx}} \overset{L}{=} \lim_{x\to\infty}\frac{1}{ce^{cx}} = \frac{1}{c}\lim_{y\to\infty} e^{-y} = 0.$$

Consequences: $\lim_{x\to 0^+} x^a \log x \overset{I}{=} \lim_{x\to\infty}\frac{-\log x}{x^a} = 0 \quad (a > 0) \quad \boxed{0\cdot\infty}$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\log x} = e^{\lim_{x\to 0^+} x\log x} = e^0 = 1 \quad \boxed{0^0}$$

$$\lim_{x\to\infty} x^{1/x} \overset{I}{=} \lim_{x\to 0^+}\left(\frac{1}{x}\right)^x = \lim_{x\to 0^+}\frac{1}{x^x} = 1. \quad \boxed{\infty^0}$$

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[Problem: try $\lim_{x\to\infty}\left[\log(2x^2+1) - \log(x^2+x+2)\right]$, $\lim_{x\to\infty}\left(\sqrt{x+1} - \sqrt{x}\right)$ $\boxed{\infty - \infty}$

Proofs :

$\boxed{(\text{V.}1)}$ Extend f, g to $[a,b)$ by setting $f(a)=0=g(a)$ so they're defined in $[a,x]$ for any $x \in (a,b)$.
Since $g'>0$ or <0, and $g(a)=0$, either $g>0$ or $g<0$ on (a,b).
Cauchy MVT applies: $\exists\, c \in (a,x)$ s.t.

$$\underbrace{\left(f(x)-\underset{0}{f(a)}\right)\underset{\overset{\neq}{0}}{g'(c)} = \left(g(x)-\underset{0}{g(a)}\right)}_{\text{by assumption}} f'(c)$$

$\Longrightarrow \dfrac{f(x)}{g(x)} = \dfrac{f'(c)}{g'(c)}$. Let $\epsilon>0$ be given.

If $\lim\limits_{c\to a}\dfrac{f'(c)}{g'(c)} = L$, then $\exists\, \delta>0$ s.t. $c\in(a,a+\delta) \Rightarrow \left|\dfrac{f'(c)}{g'(c)}-L\right| < \epsilon$.

Taking $x \in (a,a+\delta)$, $c \in (a,x) \subset (a,a+\delta) \Rightarrow \left|\dfrac{f(x)}{g(x)}-L\right| = \left|\dfrac{f'(c)}{g'(c)}-L\right| < \epsilon$. □

$\boxed{\text{Inversion}}$ Given $\epsilon>0$, $\exists\, M>0$ s.t. $t>M \Rightarrow |f(t)-L|<\epsilon$.
So $0 < x < \underbrace{\frac{1}{M}}_{\text{take this as our } \delta>0} \Longrightarrow \dfrac{1}{x} > M \Longrightarrow \left|f\left(\tfrac{1}{x}\right)-L\right| < \epsilon$. □

$\boxed{(\text{V.}2)}$ $$\lim_{x\to\infty}\frac{f(x)}{g(x)} \overset{I}{=} \lim_{y\to0^+}\frac{f(1/y)}{g(1/y)} \overset{L}{=} \lim_{y\to0^+}\frac{\cancel{-\frac{1}{y^2}}f'(\frac{1}{y})}{\cancel{-\frac{1}{y^2}}g'(\frac{1}{y})} \overset{I}{=} \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$ □

$\boxed{(\text{V.}3)}$ (This one is tricky: $\dfrac{f}{g} = \dfrac{1/g}{1/f}$ won't work.) Write $L := \lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}$.

Let $\epsilon>0$. $\exists\, \delta_1>0$ s.t. $c\in(a,a+\delta_1) \Rightarrow \left|\dfrac{f'(c)}{g'(c)}-L\right| < K\epsilon$, $K := \dfrac{1}{2(1+|L|)}$.
Fix $y\in(a,a+\delta_1)$ and let $x\in(a,y)$. By Cauchy MVT we get $c\in(x,y)$ s.t. $\dfrac{f'(c)}{g'(c)} = \dfrac{f(x)-f(y)}{g(x)-g(y)} = \dfrac{f(x)}{g(x)}\cdot h(x)$, where $h(x) := \dfrac{1-\frac{f(y)}{f(x)}}{1-\frac{g(y)}{g(x)}}$,
and so $\left|\dfrac{f(x)}{g(x)}h(x) - L\right| < K\epsilon$.

Clearly $\lim\limits_{x\to a^+} h(x) = 1$, so $\exists\, \delta_2>0$ s.t. $x\in(a,a+\delta_2) \Longrightarrow |h(x)-1| < K\epsilon$ and $|h(x)| > \frac{1}{2}$. Using the triangle inequality, $\frac{1}{2}\left|\frac{f(x)}{g(x)}-L\right| < \left|\frac{f(x)}{g(x)}-L\right|\cdot|h| =$

$$\left|\frac{f}{g}h - Lh\right| \le \left|\frac{f}{g}h - L + L - Lh\right| \le \left|\frac{f}{g}h-L\right| + |L|\cdot|h-1| < K\epsilon + |L|K\epsilon$$

$= \dfrac{\epsilon}{2(1+|L|)}\cdot(1+|L|) = \dfrac{\epsilon}{2}$. So indeed, $\lim\limits_{x\to a^+}\dfrac{f(x)}{g(x)} = L$. □

[<u>Challenge problem</u>: You are eating dinner at a circular table seating 1000, or maybe 10000. After dinner, everyone randomly changes seats for dessert. Estimate the probability that no one at the table sits next to the same person twice. [Hint: first write down the probability for n diners, then replace n by x and take a limit.]