Lecture 27 : 1^{st}-order ODEs

An ordinary differential equation (ODE) is an equation of the form

$$\boxed{\sum_{i=0}^{n} F_i(x,y)\,\frac{d^i y}{dx^i} = 0}$$

from which one tries to determine $y=f(x)$ as a function of x.

The order of the equation is the highest derivative of y occurring.

Today we'll discuss first-order $(n=1)$ ODEs, i.e.

$$F_1(x,y)\,y' + F_0(x,y)\,y = 0 \quad \rightsquigarrow \quad \boxed{y' = F(x,y).} \qquad (*)$$

Ex 1/ $y' = F(x) \rightsquigarrow y = \int_0^x F(u)\,du + C$. //

Ex 2/ $y' = ky$ (constant) with initial condition $f(0) = C$.

$(f'(x) = kf(x))$

know that $g(x) := Ce^{kx}$ is one solution; if f is another, consider $h = \frac{f}{g} = \frac{f(x)e^{-kx}}{C} \Rightarrow h' = \frac{e^{-kx}}{C}\left(f'(x) - kf(x)\right) = 0$

$\Rightarrow \underset{\text{constant}}{h \equiv h(0)} = \frac{f(0)}{g(0)} = \frac{C}{C} = 1 \Rightarrow f = g$.

This is called exponential growth or decay. Sometimes convenient to write t in lieu of x: e.g. if $k=-\mu<0$, $f(t) = Ce^{-\mu t}$ might represent the remaining mass of some radioactive substance. Its half life T is def'd by $\frac{1}{2} = \frac{f(T)}{f(0)} = e^{-\mu T} \Rightarrow T = \frac{\log 2}{\mu}$

(or $\mu = \frac{\log 2}{T}$). //

We shall call (*) <u>linear</u> if it takes the form

$$\boxed{y' + P(x)\, y = Q(x)} \quad (**)$$

on some interval $I \subset \mathbb{R}$ on which we seek solutions $y = f(x)$.

Ex 3 / Homogeneous case: $Q(x) = 0$, i.e. "homogeneous 1^{st}-order linear ODE"

$$\boxed{y' + P(x)\, y = 0} \quad (***)$$

This is the case where linear combinations of solutions are solutions.

One obvious solution is $y \equiv 0$ on I.

Assume next that $y \neq 0$ on I. Then $-P(x) = \frac{y'}{y} = \frac{d}{dx} \log |y|$.

If $a \in I$, then $\log|y| = -\int_a^x P(u)\,du + C \underset{\exp}{\Rightarrow} y = K e^{-\int_a^x P(u)\,du}$.

Given the initial value $f(a) = b$, we have $y = b e^{-\int_a^x P(u)\,du}$.

In fact, these are the only solutions. Consider (for f any solution) $h(x) = f(x) e^{A(x)}$, where $A(x) := \int_a^x P(u)\,du$. Then

$$h'(x) = f'(x) e^{A(x)} + f(x) A'(x) e^{A(x)} = e^{A(x)} \underbrace{[f' + Pf]}_{0 \text{ since } f \text{ is a solution}} = 0$$

$$\Rightarrow \underset{(\text{constant})}{h \equiv h(a)} = f(a) e^{\overbrace{A(a)}^{=0}} = f(a) \Rightarrow f(x) = f(a) e^{-A(x)}. \quad //$$

General (<u>inhomogeneous</u> or <u>nonhomogeneous</u>) case, where $Q(x) \neq 0$:

let f be any solution of (**), $A(x) = \int_a^x P(u)\,du$, and $h(x) = f(x) e^{A(x)}$.

Then $h' = (f' + Pf) e^A = Q e^A \overset{\int}{\Longrightarrow} h(x) = \int_a^x Q(u) e^{A(u)}\,du + K$

$$\Rightarrow f = h e^{-A} = K e^{-A(x)} + e^{-A(x)} \int_a^x Q(u) e^{A(u)}\,du, \quad (\dagger)$$

where $K = f(a)$ uniquely determines the constant. This also highlights the fact that solutions to inhomogeneous equations <u>all differ by solutions to the homogeneous equation (***)</u>.

Ex 4/ (Falling body with air resistance)

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I take down = $\ominus$

Let $y = v(t)$ be the vertical velocity, s = vertical position, m = mass, $a = y'$ = acceleration. We have $ma = -mg - kv$ $(k, g > 0)$

$$\Rightarrow \quad a = -g - \frac{k}{m} v \quad \Rightarrow \quad v' + \underbrace{\frac{k}{m}}_{P} v = \underbrace{-g}_{Q}. \quad \text{Writing}$$

$A(t) := \int_0^t P(u)\,du = \int_0^t \frac{k}{m}\,du = \frac{kt}{m}$, (†) gives

$$v(t) = Ce^{-\frac{kt}{m}} - e^{-\frac{kt}{m}} \int_0^t g e^{\frac{ku}{m}}\,du = Ce^{-\frac{kt}{m}} - \frac{gm}{k}, \text{ so}$$

if $v(0) = 0$ then $v(t) = \frac{gm}{k}\left(e^{-\frac{kt}{m}} - 1\right) \Rightarrow$

$$s(t) = \int_0^t v(u)\,du + s(0) = s(0) + \frac{gm^2}{k^2}\left(1 - e^{-\frac{kt}{m}}\right) - \frac{gmt}{k}$$

and $a(t) = v'(t) = -ge^{-kt/m}$. Notice that $a \to 0$ and $v \to -\frac{gm}{k}$ (limiting velocity) as $t \to \infty$. This has a rather different character from the solution for $k = 0$ (no resistance), namely $s(t) = s(0) - \frac{1}{2}gt^2$. //

Ex 5/ (Diluting a saline solution)

Given: tank w/ 100 gallons brine (κ_0 lbs. of salt/gallon)

5 gallons/min. of weaker brine (κ_1 lbs./gallon) is flowing in to tank

5 gallons/min. is draining out of the tank

(& mixing immediately)

Determine amount of salt $y = f(t)$.

Clearly $f(0) = 100\,\kappa_0$. Inflow contributes $5\kappa_1$ lbs./min. to y

Outflow contributes $-5\underbrace{\left(\frac{y}{100}\right)}_{\text{concentration at any time } t}$ lbs./min to y.

$$\Rightarrow \quad y' + \frac{1}{20} y = 5\kappa_1$$

$$\underset{(\dagger)}{\Rightarrow} \quad y = Ce^{-t/20} + e^{-t/20} \underbrace{\int_0^t 5\kappa_1 e^{u/20}\,du}_{5\kappa_1 (20e^{t/20} - 1)} = C_0 e^{-t/20} + 100\kappa_1$$

$C_0 = 100(\kappa_0 - \kappa_1)$ by setting $t = 0$

$$= 100\kappa_0 e^{-t/20} + 100\kappa_1\left(1 - e^{-t/20}\right).$$ //