Lecture 28: 2^{nd}-order linear ODEs (I)

Today we shall seek solutions to the differential equation

$$(*) \quad \boxed{y'' + ay' + by = 0}$$

on all of $\mathbb{R}$.

Ex 1/ $(a = b = 0)$ $\quad y'' = 0 \xrightarrow{\int} y' = c_1 \xrightarrow{\int} y = c_1 x + c_2$. //

Ex 2/ $(a = 0, \underset{-k^2}{\underset{\shortparallel}{b}} < 0)$ $\quad y'' = k^2 y$: both $y = e^{kx}$ and $y = e^{-kx}$ are solutions $\Rightarrow$ $y = c_1 e^{kx} + c_2 e^{-kx}$ is a solution for every $c_1, c_2 \in \mathbb{R}$. //

Ex 3/ $(a = 0, \underset{k^2}{\underset{\shortparallel}{b}} > 0)$ $\quad y'' = -k^2 y$: both $y = \cos(kx)$ & $\sin(kx)$ are solutions $\Rightarrow$ $y = c_1 \sin(kx) + c_2 \cos(kx)$ is a solution $\forall c_1, c_2 \in \mathbb{R}$. //

<u>Theorem 1</u>: These give all solutions $y = f(x)$ of equations of the form $y'' + by = 0$.

<u>Proof</u>: $\boxed{\text{Step 1}}$ Given initial conditions $f(0) = A$, $f'(0) = B$, there <u>exists</u> a solution of the form above: e.g. for $b = k^2 > 0$, we get $f(0) = c_2$ and $f'(0) = c_1 k \Rightarrow c_1 = \frac{f'(0)}{k}$.

$\boxed{\text{Step 2}}$ <u>Uniqueness</u> of solution with these initial conditions: if $g(x)$ is another, consider $h(x) = f(x) - g(x)$ (which solves the DE, with $h(0) = 0 = h'(0)$). Since $h'' = -bh$, by induction we get $h^{(n)}(0) = 0$ $(\forall n)$ $\left[\text{use } h^{(n)} = (h'')^{(n-2)} = -bh^{(n-2)}\right]$. So the Taylor

polynomials of h at 0 are all 0, so $h = T_{2n-1} + E_{2n-1} = E_{2n-1}$.
Moreover $|h^{(2n)}(x)| = |b|\,|h^{(2n-2)}(x)| = \cdots = |b|^n\,|h(x)| \Rightarrow$ if $M := \sup_{x\in[-c,c]} |h(x)|$,
then $|h^{(2n)}(x)| \le |b|^n M$ on $[-c,c]$. So (on $[-c,c]$)

$$0 \le |h(x)| = |E_{2n-1}(x)| = \frac{|h^{(2n)}(x)|}{(2n)!}|x|^{2n} \le \frac{M}{(2n)!}|\underbrace{b^{1/2}c}_{=:A}|^{2n}$$

for any n. By taking n sufficiently large, $\frac{A^{2n}}{(2n)!}$ may be made arbitrarily small,† so $h(x) \equiv 0$ on $[-c,c]$ hence (c was arbitrary) on all of $\mathbb{R}$. $\square$

Consider now the characteristic equation

$$r^2 + ar + b = 0$$

associated to $(*)$: it has roots $r_\pm = \frac{-a \pm \sqrt{d}}{2}$, where $d = a^2 - 4b$ is called the discriminant.

Theorem 2: The solutions of $(*)$ are given by $y = e^{-\frac{a}{2}x} g(x)$, where $z = g(x)$ solves $z'' - \frac{d}{4} z = 0$.

- $c_1 + c_2 x$ if $d = 0$
- $c_1 e^{\frac{\sqrt{d}}{2}x} + c_2 e^{-\frac{\sqrt{d}}{2}x}$ if $d > 0$
- $c_1 \sin\left(\frac{\sqrt{-d}}{2}x\right) + c_2 \cos\left(\frac{\sqrt{-d}}{2}x\right)$ if $d < 0$.

Proof: Let $f(x) = e^{-\frac{a}{2}x} g(x)$. We have

$$f' = -\frac{a}{2}e^{-\frac{a}{2}x} g + e^{-\frac{a}{2}x} g' = e^{-\frac{a}{2}x}\left[g' - \frac{a}{2}g\right]$$

$$f'' = -\frac{a}{2}e^{-\frac{a}{2}x}\left[g' - \frac{a}{2}g\right] + e^{-\frac{a}{2}x}\left[g'' - \frac{a}{2}g'\right] = e^{-\frac{a}{2}x}\left[g'' - ag' + \frac{a^2}{4}g\right].$$

So $f'' + af' + bf = 0 \iff e^{-\frac{a}{2}x}\left\{\left(g'' \cancel{-ag'} + \frac{a^2}{4}g\right) + a\left(\cancel{g'} - \frac{a}{2}g\right) + bg\right\} = 0$

$\iff g'' + \underbrace{\left(b - \frac{a^2}{4}\right)}_{-\frac{d}{4}} g = 0.$ $\square$

($e^{-\frac{a}{2}x}$ never 0)

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† Let $k = \lfloor A \rfloor$. Then $\frac{A}{k+1} < 1$ & $\frac{A^m}{m!} \le \frac{A^k}{k!} \cdot \left(\frac{A}{k+1}\right)^{m-k}$, and $\left(\frac{A}{k+1}\right)^{m-k} \to 0$ as $m \to \infty$.

In your HW you'll show that given two solutions of $(*)$ which are not merely constant multiples of each other, their linear combinations give all solutions to $(*)$. The idea is to use the Wronskian of these two solutions.

Definition: The Wronskian of two functions $u_1(x), u_2(x)$ is

$$W(x) := u_1 u_2' - u_1' u_2 .$$

Some properties (problems 21 & 22 on pp. 328-9):

- $W' = \cancel{u_1' u_2'} + u_1 u_2'' - u_1'' u_2 - \cancel{u_1' u_2'} = u_1 u_2'' - u_1'' u_2 .$
- Let I be an interval on which $u_1 \neq 0$. Then $W \equiv 0 \iff u_1$ & u_2 are $\underbrace{\text{constant multiples of each other}}_{\text{"dependent"}}$.
  $\left[\text{Wherever } u_1 \neq 0,\ \left(\frac{u_2}{u_1}\right)' = \frac{u_1 u_2' - u_1' u_2}{u_1^2} = \frac{W}{u_1^2} = 0 .\right]$
- If u_1, u_2 solve $y'' + ay' + by = 0$ (with u_1 not identically 0),
  $W' = u_1 u_2'' - u_1'' u_2 = -a u_1 u_2' - \cancel{b u_1 u_2} + a u_1' u_2 + \cancel{b u_1 u_2} = -aW$
  $\Rightarrow W(x) = W(0)\, e^{-ax}.$
- In the last situation, suppose $W(0) \neq 0$. Then $W(x) \neq 0$ for all x, hence u_1 & u_2 are independent (not constant multiples of each other on $\mathbb{R}$).
  If $W(0) = 0$, then $W(x) \equiv 0$, but it doesn't follow for arbitrary functions that u_1 & u_2 are dependent (consider $x|x|$ & x^2). What saves us here is that u_1 & u_2 are linear combinations of $e^{-a/2 x}$ & $x e^{-a/2 x}$, $e^{-(a/2)x} \sin\left(\frac{\sqrt{-d}}{2} x\right)$ & $e^{-(a/2)x} \cos\left(\frac{\sqrt{-d}}{2} x\right)$, or $e^{-(a/2)x} e^{\sqrt{d}/2\, x}$ & $e^{-(a/2)x} e^{-\sqrt{d}/2\, x}$. So from $W(x) \equiv 0$ we get on some interval I that $\alpha u_1 + \beta u_2 = 0$; that is, some linear combination of (say) $\sin(kx)$ & $\cos(kx)$ is 0 on I. The only way this happens is if it is $0 \sin kx + 0 \cos kx$, hence zero on all of $\mathbb{R}$.