Lecture 29: Complex numbers

A field is a set $\mathbb{F}$ with:

- 2 commutative, associative binary operations $+, \cdot$
- identity elements "0" for $+$, "1" for $\cdot$
- inverses: $(-a)$ for $+$, $\underbrace{\left(\frac{1}{a}\right) \text{ for } \cdot}_{\text{if } a \neq 0}$
- distributive law.

$\mathbb{F}$ is algebraically closed if for every polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

with each $a_i \in \mathbb{F}$, there exists a solution (for z) in $\mathbb{F}$.

Since $z^2+1=0$ has no real solution, the field $\mathbb{R}$ of real numbers is not algebraically closed.

Define the field of complex numbers by

$$\mathbb{C} := \{ a+bi \mid a, b \in \mathbb{R} \} \quad (\cong \mathbb{R}^2 \text{ as a set})$$

Cartesian form

with

- addition given by $(a_1+b_1 i) + (a_2+b_2 i) := (a_1+a_2) + (b_1+b_2)i$
  (and additive inverse $-(a+bi) := (-a) + (-b)i$)
- multiplication given by
  $$(a_1+b_1 i)\cdot(a_2+b_2 i) := (a_1 a_2 \underbrace{- b_1 b_2}_{}) + (a_1 b_2 + a_2 b_1)i$$
  treat i as "$\sqrt{-1}$", i.e. as a formal solution to $z^2 = -1$

For the mult. inverse,

note that $(a+bi)(a-bi) = a^2+b^2$, so that

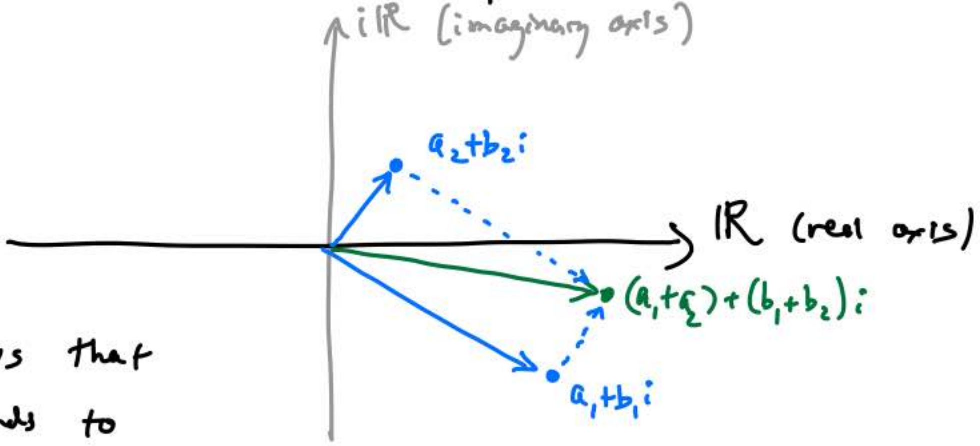
$$(a+bi)\underbrace{\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i\right)}_{=: \frac{1}{a+bi} \text{ or } (a+bi)^{-1}} = 1.$$

$=: \frac{1}{a+bi}$ or $(a+bi)^{-1}$, defined if a, b not both zero.

The map $\mathbb{R} \hookrightarrow \mathbb{C}$ presents $\mathbb{R}$ as a subfield of $\mathbb{C}$,
$r \longmapsto r + 0i$ (which we just write "r")

much as $\mathbb{Q}$ is a subfield of $\mathbb{R}$. (This map respects $+, \cdot$.)

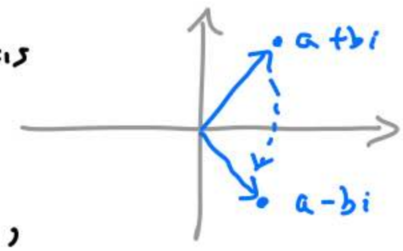
We visualize $\mathbb{R} \subset \mathbb{C}$ via the picture



which also shows that addition corresponds to adding ordered pairs or "vectors" (arrows) in $\mathbb{R}^2$. We can also define

- complex conjugation := flip about $\mathbb{R}$-axis

  $\overline{a+bi} := a - bi$



  This respects $+, \cdot$: given $\alpha_1, \alpha_2 \in \mathbb{C}$ ($\alpha_j = a_j + b_j i$),

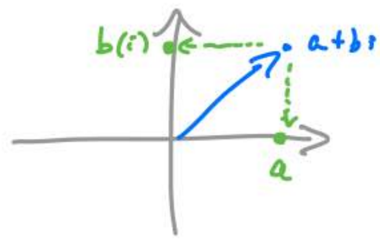
  $\overline{\alpha_1 + \alpha_2} = \overline{\alpha_1} + \overline{\alpha_2}$ and $\overline{\alpha_1 \cdot \alpha_2} = \overline{\alpha_1} \cdot \overline{\alpha_2}$ (check).

  We also have $\overline{\overline{\alpha}} = \alpha$.

- real & imaginary parts : given $\alpha = a + bi \in \mathbb{C}$,

  set $\mathrm{Re}(\alpha) := a = \frac{1}{2}(\alpha + \overline{\alpha})$

  $\mathrm{Im}(\alpha) := b = \frac{1}{2i}(\alpha - \overline{\alpha})$ (check)



- modulus (or absolute value) : $|\alpha| := \sqrt{a^2+b^2}$ = distance from 0 to α

  Notice that $\alpha\overline{\alpha} = (a+bi)\cdot(a-bi) = a^2 + b^2 + (ab - ab)\,i = |\alpha|^2$

  $\Rightarrow \alpha \cdot \frac{\overline{\alpha}}{|\alpha|^2} = 1 \Rightarrow \alpha^{-1} = \overline{\alpha}/|\alpha|^2$. This gives a simple way of thinking about (even visualizing) the mult. inverse. But in practice (to compute inverses & quotients) we just write

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\,i.$$

Properties of modulus: $|\bar{\alpha}| = |\alpha|$ is obvious; and

$|\alpha\beta| = |\alpha||\beta|$ is easy: $|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta}) = (\alpha\bar{\alpha})(\beta\bar{\beta}) = |\alpha|^2|\beta|^2$.

More interesting is the triangle inequality: $|\alpha+\beta| \le |\alpha| + |\beta|$.

[Proof: For $z = x+iy \in \mathbb{C}$, $\mathrm{Re}(z) = x \le |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z|$.

Now $|\alpha+\beta|^2 = (\alpha+\beta)\overline{(\alpha+\beta)} = \alpha\bar{\alpha} + \beta\bar{\beta} + \alpha\bar{\beta} + \bar{\alpha}\beta$ i.e. $\overline{\alpha\bar{\beta}}$

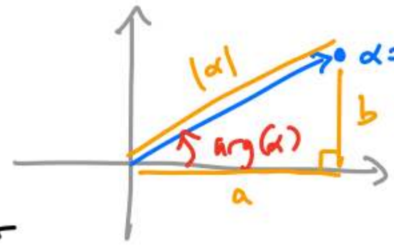
$= |\alpha|^2 + |\beta|^2 + 2\,\mathrm{Re}(\alpha\bar{\beta})$ $\le 2|\alpha\bar{\beta}| = 2|\alpha||\bar{\beta}| = 2|\alpha||\beta|$

$\le |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| = (|\alpha| + |\beta|)^2$. Take $\sqrt{\ }$.]

- argument of a complex number: $\arg(\alpha) :=$ angle the segment from 0 to α makes w/ $\mathbb{R}$-axis

Notice that if $a > 0$ then

$$\arg(a+bi) = \arctan\left(\frac{b}{a}\right).$$



If $a < 0$ then you have to add or subtract π. I should also mention that $\arg(\alpha)$ is only defined up to integer multiples of 2π; that is, for our purposes here $2\pi \equiv 0$. (This is called working modulo 2π or $2\pi\mathbb{Z}$.)

Upshot: Writing $\theta := \arg(\alpha)$ and $r = |\alpha|$ then $a = r\cos\theta$ and $b = r\sin\theta$, hence $\alpha = r(\cos\theta + i\sin\theta)$. (polar form, v.1)

Here is the key property of arguments of complex numbers:

Proposition: If α_1, α_2 have arguments θ_1, θ_2 then $\alpha_1\alpha_2$ has argument $\theta_1 + \theta_2$ (mod 2π). (and so α_1/α_2 has argument $\theta_1 - \theta_2$)

Proof: Since $|\alpha_1\alpha_2| = |\alpha_1||\alpha_2|$, this reduces to checking that

$$(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2).$$

But this is just the addition laws $\begin{cases} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2) \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2). \end{cases}$ $\square$

Ex / Find $\sqrt{3+4i}$.

Let's think about square roots in general: if $\overset{a+bi}{\alpha} = \overset{\sqrt{a^2+b^2}}{r}\left(\cos \overset{\arctan\frac{b}{a}}{\theta} + i\sin\theta\right)$, then the Proposition tells us that $\left(r^{1/2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right)^2 = \alpha$ hence that

$$\alpha^{1/2} = r^{1/2}\cos\frac{\theta}{2} + ir^{1/2}\sin\frac{\theta}{2} = r^{1/2}\sqrt{\frac{1+\cos\theta}{2}} + ir^{1/2}\sqrt{\frac{1-\cos\theta}{2}}$$

$$= \sqrt{\frac{r+r\cos\theta}{2}} + i\sqrt{\frac{r-r\cos\theta}{2}} = \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.$$

(Actually, $-\alpha^{1/2}$ will also be a square root too.) Applying this formula,

$$\sqrt{3+4i} = \sqrt{\frac{\sqrt{3^2+4^2}+3}{2}} + i\sqrt{\frac{\sqrt{3^2+4^2}-3}{2}} = \sqrt{\frac{5+3}{2}} + i\sqrt{\frac{5-3}{2}}$$

$$= \sqrt{4} + i\sqrt{1} = 2+i.$$

Check: $(2+i)^2 = 2^2 + i^2 + 2i + 2i = 3+4i$. //

You may be wondering still, since I mentioned that $\mathbb{R}$ was <u>not</u> algebraically closed, about $\mathbb{C}$:

<u>Fundamental Theorem of Algebra</u>: $\mathbb{C}$ is algebraically closed.

We can't prove this right now, but I should point out that for a quadratic equation the existence of complex solutions is clear:

$$Az^2 + Bz + C = 0 \quad (A, B, C \in \mathbb{C})$$

$$\rightsquigarrow z = \frac{-B \pm \sqrt{B^2-4AC}}{2A} \quad \text{\underline{always} exists in } \mathbb{C}.$$

Now suppose $A, B, C \in \mathbb{R}$.

If one solution is non-real (b/c $B^2 - 4AC < 0$), then so is the other, and the two solutions are <u>conjugate</u>!

That is, $Az^2 + Bz + C = A(z-\alpha)(z-\bar{\alpha}) = A(z^2 - \underbrace{(\alpha+\bar{\alpha})}_{2\,\mathrm{Re}(\alpha)}z + \underbrace{\alpha\bar{\alpha}}_{|\alpha|^2})$

— so conversely, if roots are <u>conjugate</u> the equation has real coefficients.