Lecture 3: Sets of real numbers

Set notation: Sets are collections of elements.

- "roster notation": $A = \{2, 3, 4, 5, 6\}$ (list the elements)

  write $2 \in A$ to say "2 is an element of A"

  write $\{2\} \subseteq A$ to say "$\{2\}$ is a subset of A"

  (other subsets of A: $\{2,4\}$, $\{3,5,6\}$, A itself, $\emptyset$)

- "predicate notation": $A = \{ \underset{\substack{\uparrow \\ \text{domain}}}{x \in \mathbb{Z}} \mid \underbrace{2 \le x \le 6}_{\text{predicate (a statement that can hold or not for } x \in \text{domain)}} \}$

  this requires A to be a subset of the "domain". Some common sets of numbers:

  $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rationals), $\mathbb{R}$ (reals), $\mathbb{C}$ (complex #s)

- building sets from other sets: given sets A and B,

  $A \cup B$ = union = $\{x \mid x \in A \text{ or } x \in B\}$

  $A \cap B$ = intersection = $\{x \mid x \in A \text{ and } x \in B\}$

  $A - B$ or $A \setminus B$ = difference = $\{x \mid x \in A \text{ and } x \notin B\}$

  Can do the same for a collection (or "family") $\mathcal{F} = \{A_1, \dots, A_n\}$:

  $$\bigcup_{A \in \mathcal{F}} A = \bigcup_{k=1}^{n} A_k \quad , \quad \bigcap_{A \in \mathcal{F}} A = \bigcap_{k=1}^{n} A_k$$

The real numbers

These are defined axiomatically in the text:

- Field axioms (commutativity, associativity, distributivity, negatives & inverses [except for $\frac{1}{0}$!], etc.) — true for $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

- Order axioms : existence of a subset of positive numbers closed under addition and multiplication, such that every x has $x=0$, $x>0$ or $x<0$. (Define $a<b$ if $b-a$ is positive.) — true for $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Here's a consequence — TRANSITIVE LAW: if $a<b$ & $b<c$, then $a<c$.

Proof: $b-a>0$ and $c-b>0 \overset{(\mathbb{R}^+ \text{ closed under } +)}{\Longrightarrow}$ their sum is >0, i.e. $(b-a)+(c-b) = c-a > 0$. $\square$

- The least-upper-bound axiom : (this one is only true for $\mathbb{R}$) Suppose $S \subset \mathbb{R}$ is nonempty & bounded above : i.e., $\exists\, b \in \mathbb{R}$ such that $s \le b$ for every $s \in S$. Then S has a least upper bound — that is, $B \in \mathbb{R}$ such that (i) B is an upper bound for S
  (ii) no number less than B is an upper bound for S.

"supremum" (or "least upper bound")

We write $\sup S := B$

"upper bound"

Similarly, if S is bounded below, then it has a greatest lower bound or infimum $\inf S \ \left(:= -\sup(-S)\right)$.

These axioms define $\mathbb{R}$; one then defines

- the positive numbers $\mathbb{P}$ as in lecture 1 (basically, $1, 1+1, 1+1+1, \ldots$)
- the integers $\mathbb{Z} = \mathbb{P} \cup \{0\} \cup -\mathbb{P}$
- the rationals $\mathbb{Q} := \left\{ \frac{a}{b} \;\middle|\; a,b \in \mathbb{Z} \text{ and } b \neq 0 \right\}$

Of course, $\mathbb{P} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Archimedean Property : If $x \in \mathbb{R}^+$, $y \in \mathbb{R}$, then $\exists\, n \in \mathbb{P}$ s.t. $nx > y$.

Proof : $\mathbb{P}$ is not bounded above (otherwise, have $\beta = \sup \mathbb{P}$ and $m \in \mathbb{P}$ with $m > \beta - 1 \Rightarrow m+1 > \beta$ ※). So there must be an element of $\mathbb{P}$ bigger than y/x. $\square$

Ex/ Let $\mathcal{R} := \{x \in \mathbb{R} \mid x^2 < 2\}$. This is bounded above (by, say, 2) & nonempty, so has a least upper bound $\mathcal{B} := \sup \mathcal{R}$. There are only 3 possibilities: $\mathcal{B}^2 < 2$, $\mathcal{B}^2 > 2$, or $\mathcal{B}^2 = 2$. If we can rule out the first two, then the third holds, and $\mathcal{B}$ is a square root of 2.

- suppose $\mathcal{B}^2 > 2$. Let $c := \mathcal{B} - \frac{\mathcal{B}^2 - 2}{2\mathcal{B}} = \frac{\mathcal{B} + \frac{2}{\mathcal{B}}}{2}$,

  so that $0 < c < \mathcal{B}$ and

  $$c^2 = \mathcal{B}^2 - (\mathcal{B}^2 - 2) + \frac{(\mathcal{B}^2-2)^2}{4\mathcal{B}^2} = 2 + \frac{(\mathcal{B}^2-2)^2}{4\mathcal{B}^2} > 2$$

  $\Rightarrow$ c is an upper bound for $\mathcal{R}$

  $\underset{(\mathcal{B}\text{ least UB})}{\Rightarrow}$ $\mathcal{B} \leq c$ $\lightning$

- suppose $\mathcal{B}^2 < 2$. Let $c \in \mathbb{R}^+$ be less than $\mathcal{B}$ and $\frac{2 - \mathcal{B}^2}{3\mathcal{B}}$, so that $(\mathcal{B} + c)^2 < \mathcal{B}^2 + 3\mathcal{B}c < \mathcal{B}^2 + (2 - \mathcal{B}^2) = 2$

  $\Rightarrow$ $\mathcal{B} + c \in \mathcal{R}$

  $\underset{(\mathcal{B}\text{ UB})}{\Rightarrow}$ $\mathcal{B} + c \leq \mathcal{B}$ $\Rightarrow$ $c \leq 0$ $\lightning$ //

<u>Upshot</u>: "$\sqrt{2}$" exists in $\mathbb{R}$. But not in $\mathbb{Q}$:

Ex/ Let $S := \{x \in \mathbb{Q} \mid x^2 < 2\}$. I claim that S does <u>NOT</u> have a least upper bound. We need 2 facts:

- "$\sqrt{2}$ is irrational": suppose there was a rational number $\frac{a}{b} \in \mathbb{Q}$ $(a, b \in \mathbb{P})$ with $\left(\frac{a}{b}\right)^2 = 2$. We may assume that a or b is odd, since otherwise we can cancel powers of 2 until this is true (why?). Now $a^2 = 2b^2 \Rightarrow a$ can't be odd $\Rightarrow a$ even $= 2c$, $c \in \mathbb{P}$ $\Rightarrow 4c^2 = 2b^2 \Rightarrow 2c^2 = b^2 \Rightarrow b$ even. Contradiction.

- If $p, r \in \mathbb{R}$ and $p < r$, then $\exists\ q \in \mathbb{Q}$ with $p < q < r$.

  I'll prove this tomorrow: it's a consequence of the Archimedean property together with the "well-ordering principle".

- So now, let $r \in \mathbb{Q}$ be a least upper bound for S, and put $p = \sqrt{2}$. We can't have $r = p$ (since $p \notin \mathbb{Q}$). Suppose $r < p$; then $\exists\ t \in \mathbb{Q}$ between them, so that $t^2 < p^2 = 2 \Rightarrow t \in S$ but $r < t$, impossible since r is an upper bound for S. So we are left with $r > p$; but then any $s \in \mathbb{Q}$ between them is an upper bound for S. Contradiction. //

Upshot: $\mathbb{Q}$ does not satisfy the least upper bound axiom.
(It isn't "dense" enough.)

Properties of inf & sup (Apostol, pp. 26-28)

Lemma: If $a, x, y \in \mathbb{R}$ satisfy $a \le x \le a + \frac{y}{n}$ $\forall n \in \mathbb{P}$, then $x = a$.

Proof: By the Archimedean property, if $x > a$ then $\exists\ n \in \mathbb{P}$ s.t. $n(x-a) > y$, i.e. $x > a + \frac{y}{n}$. ※ Clearly $x < a$ is false, so $x = a$. □

Property I: If $S \subset \mathbb{R}$ has UB (LB), and $h \in \mathbb{R}^+$, then $\exists\ x \in S$ with $x > \sup S - h$ ($x < \inf S + h$).

Proof: Otherwise $\sup S - h$ is an UB for S, impossible b/c $\sup S$ is least. □

Property II: Given $A, B \subset \mathbb{R}$ with UB, $C := \{a+b \mid a \in A, b \in B\}$, we have $\sup C = \sup A + \sup B$. (Same for LB / inf)

Proof: Since $\sup A + \sup B$ is an UB for C (why?),

$\sup C \le \sup A + \sup B$. By property I, for any $n \in \mathbb{P}$ $\exists \begin{cases} a \in A \\ b \in B \end{cases}$ s.t. $\begin{cases} a > \sup A - \frac{1}{n} \\ b > \sup B - \frac{1}{n} \end{cases} \Longrightarrow$

$$\sup A + \sup B < \underbrace{a+b}_{\in C} + \frac{2}{n} \le \sup C + \frac{2}{n}.$$

Now apply the Lemma! $\square$

<u>Property III</u>: If $S, T \subset \mathbb{R}$ are nonempty and $\forall s \in S, t \in T$ we have $s < t$, then $\sup S \le \inf T$.

<u>Proof</u>: Every $t \in T$ is an UB for S, so $\underset{\text{(exists!)}}{\sup S} \le t$ $(\forall t \in T)$

$\Longrightarrow \sup S$ is a LB for $T \underset{\left(\substack{\inf T \\ \text{is greatest LB}}\right)}{\Longrightarrow} \sup S \le \inf T$. $\square$